A CONFORMALLY INVARIANT SPHERE THEOREM IN FOUR DIMENSIONS

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0. Introduction

Under what conditions on the curvature can we conclude that a smooth, closed Riemannian manifold is diffeomorphic (or homeomorphic) to the sphere? A result which addresses this question is usually referred to as a *sphere theorem*, and the literature abounds with examples (see Chapter 11 of [Pe] for a brief survey).

In this paper we concentrate on four dimensions, where Freedman's work is obviously very influential. For example, any curvature condition which implies the vanishing of the de Rham cohomology groups $H^1(M^4, \mathbf{R})$ and $H^2(M^4, \mathbf{R})$ will, by Freedman's result ([Fr]), imply that M^4 is covered by a homeomorphism sphere. At the same time, there are some very interesting results which characterize the *smooth* four-sphere.

An example of particular importance to us is the work of Margerin ([Ma2]), in which he formulated a notion of "weak curvature pinching." To explain this we will need to establish some notation. Given a Riemannian four-manifold (M^4, g), let $Riem$ denote the curvature tensor, W the Weyl curvature tensor, Ric the Ricci tensor, and R the scalar curvature. The usual decomposition of $Riem$ under the action of $O(4)$ can be written

$$Riem = W + \tfrac{1}{2}E \owedge g + \tfrac{1}{24}Rg \owedge g, \tag{0.1}$$

where $E = Ric - \frac{1}{4}Rg$ is the trace-free Ricci tensor and $\owedge$ denotes the Kulkarni-Nomizu product. If we let $Z = W + \frac{1}{2}E \owedge g$, then

$$Riem = Z + \tfrac{1}{24}Rg \owedge g.$$

Note that (M^4, g) has constant curvature if, and only if, $Z \equiv 0$. We now define the scale-invariant "weak pinching" quantity

$$WP \equiv \frac{|Z|^2}{R^2} = \frac{|W|^2 + 2|E|^2}{R^2} \tag{0.2}$$

where $|Z|^2 = Z_{ijkl}Z^{ijkl}$ denotes the norm of Z viewed as a $(0, 4)$–tensor.

Margerin's main result states that if $R > 0$ and $WP < \frac{1}{6}$, then M^4 is diffeomorphic to either S^4 or $\mathbf{R}P^4$. Moreover, this "weak pinching" condition is *sharp*: The spaces $(\mathbf{C}P^2, g_{FS})$ and $(S^3 \times S^1, g_{prod.})$ both have $R > 0$ and $WP \equiv \frac{1}{6}$. Indeed, using a holonomy reduction argument, Margerin also proved the converse, in the sense that these manifolds (and any quotients) are characterized by the property that $WP \equiv \frac{1}{6}$.

Margerin's proof relied on an important tool in the subject of sphere theorems, namely, Hamilton's Ricci flow. In fact, previously Huisken ([Hu]) and Margerin ([Ma1]) independently had used the Ricci flow to prove a similar pinching result, but with a slightly worse constant. In addition, Hamilton ([Ha]) had used his flow to study four-manifolds with positive curvature operator. As Margerin points out in his introduction, there is no relation between weak

pinching and positivity of the curvature operator; indeed, weak pinching even allows for some negative sectional curvature.

On the homeomorphism level, both Margerin's and Hamilton's curvature assumptions already imply that the underlying manifold is covered by a homeomorphism sphere. If (M^4, g) has positive curvature operator, then the classical Bochner theorem implies that $H^1(M^4, \mathbf{R}) = H^2(M^4, \mathbf{R}) = 0$. As we observed above, Freedman's work then gives the homeomorphism type of the cover of M^4. For manifolds satisfying $WP < \frac{1}{6}$ the vanishing of harmonic forms is less obvious, but does follow from [Gu].

One drawback to the sphere theorems described above is that they require one to verify a *pointwise* condition on the curvature. In contrast, consider the (admittedly much simpler) case of surfaces. For example, if the Gauss curvature of the surface (M^2, g) satisfies $\int K dA > 0$, then M^2 is diffeomorphic to S^2 or $\mathbf{R}P^2$. In addition to this topological classification, the uniformization theorem implies that (M^2, g) is conformal to a surface of constant curvature, which is then covered isometrically by S^2. Therefore, in two dimensions one has a "sphere theorem" which only requires one to check an *integral* condition on the curvature.

Our goal in this paper is to generalize this situation, by showing that the smooth four-sphere is also characterized by an integral curvature condition. As we shall see, our condition has the additional properties of being *sharp* and *conformally invariant*. Although there are different—though equivalent—ways of stating our main result, the simplest version involves the Weyl curvature and the Yamabe invariant:

Theorem A. *Let (M^4, g) be a smooth, closed four-manifold for which*

  (i) *the Yamabe invariant $Y(M^4, g) > 0$, and*

  (ii) *the Weyl curvature satisfies*

$$\int_{M^4} |W|^2 dvol < 16\pi^2 \chi(M^4). \tag{0.3}$$

Then M^4 is diffeomorphic to either S^4 or $\mathbf{R}P^4$.

Remarks.

  1. Recall that the Yamabe invariant is defined by

$$Y(M^4, g) = \inf_{\tilde{g} \in [g]} vol(\tilde{g})^{-\frac{1}{2}} \int_{M^4} R_{\tilde{g}} dvol_{\tilde{g}},$$

  where $[g]$ denotes the conformal class of g. Positivity of the Yamabe invariant implies that g is conformal to a metric of strictly positive scalar curvature.

  2. In the statement of Theorem A, the norm of the Weyl tensor is given by $|W|^2 = W_{ijkl}W^{ijkl}$; i.e., the usual definition when W is viewed as a section of $\otimes^4 T^* M^4$.

However, if one views W as a section of $End(\Lambda^2(M^4))$, then the convention is $|W|^2 = \frac{1}{4}W_{ijkl}W^{ijkl}$. This can obviously lead to confusion not only when comparing formulas from different sources, but also when the Weyl tensor is interpreted in different ways in the same paper (which will be the case here). To avoid this problem, our convention will be to denote the $(0,4)$-norm using $|\cdot|$, and the $End(\Lambda^2(M^4))$-norm by $\|\cdot\|$, which has the added advantage of emphasizing how we are viewing the tensor in question. We should note that some authors (for example, Margerin) avoid this confusion by just defining an isomorphism between $\otimes^4 T^*M^4$ and $End(\Lambda^2(M^4))$ which induces the same norm on both. But in our case, we will adopt the usual identification: if $Z \in \otimes^4 T^*M^4$, then we identify this with $Z \in End(\Lambda^2(M^4))$ by defining $Z(\omega)_{ij} = \frac{1}{2}Z_{ij}{}^{kl}\omega_{kl}$.

3. That Theorem A relies on an integral curvature condition indicates the possibility one could attempt to formulate a version which imposed much weaker regularity assumptions on the metric. For example, in [CMS], Cheeger, Müller, and Schrader defined a notion of curvature on piecewise flat spaces, which also allowed a generalization of the Chern–Gauss–Bonnet formula.

By appealing to the Chern–Gauss–Bonnet formula, it is possible to replace (0.3) with a condition which does not involve the Euler characteristic. Since

$$8\pi^2 \chi(M^4) = \int_{M^4} \left(\tfrac{1}{4}|W|^2 - \tfrac{1}{2}|E|^2 + \tfrac{1}{24}R^2\right) dvol, \tag{0.4}$$

Theorem A is equivalent to

Theorem A′. *Let (M^4, g) be a smooth, closed manifold for which*

(i) *the Yamabe invariant $Y(M^4, g) > 0$, and*

(ii) *the curvature satisfies*

$$\int_{M^4} \left(-\tfrac{1}{2}|E|^2 + \tfrac{1}{24}R^2 - \tfrac{1}{4}|W|^2\right) dvol > 0. \tag{0.5}$$

Then M^4 is diffeomorphic to either S^4 or $\mathbf{R}P^4$.

Formulating the result of Theorem A in this manner allows us to explain the connection with the work of Margerin. This connection relies on recent work ([CGY1],[CGY2]) in which we established the existence of solutions to a certain fully nonlinear equation in conformal geometry. The relevance of this PDE work to the problem at hand is explained in Section 1. Simply put, the results of [CGY1] and [CGY2] allow us to prove that under the hypotheses of Theorem A′, there is a conformal metric for which the integrand in (0.5) is *pointwise* positive. That is, through a conformal deformation of metric, we are able to pass from positivity in an average sense to pointwise positivity. Now, any metric for which the integrand in (0.5) is positive must satisfy

$$|W|^2 + 2|E|^2 < \tfrac{1}{6}R^2,$$

by just rearranging terms. Note that this implies in particular that $R > 0$. Dividing by R^2, we conclude that

$$WP = \frac{|W|^2 + 2|E|^2}{R^2} < \frac{1}{6}.$$

The conclusion of the theorem thus follows from Margerin's work.

As we mentioned above, Theorem A is *sharp*. By this we mean that we can precisely characterize the case of equality:

Theorem B. *Let (M^4, g) be a smooth, closed manifold which is not diffeomorphic to either S^4 or $\mathbf{R}P^4$. Assume in addition that*

(i) *the Yamabe invariant $Y(M^4, g) > 0$,*

(ii) *the Weyl curvature satisfies*

$$\int_{M^4} |W|^2 dvol = 16\pi^2 \chi(M^4). \tag{0.6}$$

Then one of the following must be true:

1 *(M^4, g) is conformal to $\mathbf{C}P^2$ with the Fubini-Study metric g_{FS}, or*

2 *(M^4, g) is conformal to a manifold which is isometrically covered by $S^3 \times S^1$ endowed with the product metric $g_{prod.}$.*

The proof of Theorem B relies on a kind of vanishing result, in a sense which we now explain. Suppose (M^4, g) satisfies that hypotheses of Theorem B. If there is another metric in a small neighborhood of g for which the L^2-norm of the Weyl tensor is smaller; i.e.,

$$\int_{M^4} |W|^2 dvol < 16\pi^2 \chi(M^4),$$

then by Theorem A we would conclude that M^4 is diffeomorphic to either S^4 or $\mathbf{R}P^4$. This, however, contradicts one of the assumptions of Theorem B. Therefore, for every metric in some neighborhood of g,

$$\int_{M^4} |W|^2 dvol \geq 16\pi^2 \chi(M^4).$$

Consequently, g is a critical point (actually, a local minimum) of the *Weyl functional* $g \mapsto \int |W|^2 dvol$. The gradient of this functional is called the *Bach tensor*, and we will say that critical metrics are *Bach–flat*. Note that the conformal invariance of the Weyl tensor implies that Bach-flatness is a conformally invariant property. In fact, the Bach tensor is conformally invariant, [De].

Theorem B is then a corollary of the following classification of Bach-flat metrics:

Theorem C. *Let (M^4, g) be a smooth, closed manifold which is not diffeomorphic to either S^4 or $\mathbf{R}P^4$. Assume in addition that*

(i) *(M^4, g) is Bach-flat,*

(ii) *the Yamabe invariant $Y(M^4, g) > 0$,*

(iii) *the Weyl curvature satisfies*

$$\int_{M^4} |W|^2 dvol = 16\pi^2 \chi(M^4). \tag{0.6}$$

Then one of the following must be true:

1 *(M^4, g) is conformal to $\mathbf{C}P^2$ with the Fubini-Study metric g_{FS}, or*

2 *(M^4, g) is conformal to a manifold which is isometrically covered by $S^3 \times S^1$ endowed with the product metric $g_{prod.}$.*

In local coordinates, the Bach tensor is given by

$$B_{ij} = \nabla^k \nabla^\ell W_{kij\ell} + \frac{1}{2} R^{k\ell} W_{kij\ell}. \tag{0.7}$$

Thus, Bach-flatness implies that the Weyl tensor lies in the kernel of a second order differential operator. At the same time, appealing once more to the results of [CGY1] and [CGY2], we can prove that a manifold satisfying the hypotheses of Theorem C is conformal to one for which the integrand in (0.5) is identically zero. In section 4 we show how these facts, along with a complicated Lagrange-multiplier argument, leads to the classification in Theorem C.

Actually, the proof of Theorem C is the most technically demanding aspect of the present paper. First, there is a long calculation to derive an integral identity for the covariant derivative of the self-dual and anti-self-dual parts of the Weyl tensor of a Bach-flat metric. In addition to the algebraic difficulties of analyzing the curvature terms which arise in this identity, there are delicate analytic issues. For example, the conformal metric we construct based on the work of [CGY1] and [CGY2] may not be regular on all of M^4. Indeed, if it were known to be smooth, then we could appeal to the classification of metrics with $WP \equiv \frac{1}{6}$ done by Margerin. These regularity problems are the price we pay, so to speak, for passing from integral to pointwise conditions.

We conclude the introduction with a note about the organization of the paper. In Section 1 we develop the necessary PDE material from [CGY1] and [CGY2]. Most of the results are fairly straightforward generalizations of our earlier work. In Section 2 we show how the results of Section 1 and the work of Margerin can be combined to prove Theorem A. Then, in Section 3 we lay the groundwork for the proof of Theorems B and C by deriving various identities for the curvature of Bach-flat metrics. In Section 4 we use these identities, along with an existence result from Section 1, to derive a key inequality for a certain polynomial in the curvature.

Analyzing this inequality leads us to consider a difficult Lagrange-multiplier problem, whose resolution gives the classification in the statement of Theorem C.

1. An existence result

In this section we prove an existence result in conformal geometry which allows us to pass from the integral conditions of Theorems A–C to their pointwise counterparts. As we indicated in the Introduction, this result is based on the work in [CGY1] and [CGY2].

To place this result in its proper context, we begin by introducing some notation. Given a Riemannian four–manifold (M^4, g), the *Weyl–Schouten* tensor is defined by

$$A = Ric - \frac{1}{6}Rg$$

In terms of the Weyl–Schouten tensor, the decomposition (0.1) can be written

$$\text{Riem} = W + \frac{1}{2}A \owedge g. \tag{1.0}$$

This splitting of the curvature tensor induces a splitting of the Euler form. To describe this, we introduce the elementary symmetric polynomials $\sigma_\kappa : \mathbf{R}^n \to \mathbf{R}$,

$$\sigma_\kappa(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_\kappa}.$$

For a section $\mathcal{S}$ of $\text{End}(TM^4)$ — or, equivalently, a section of $T^\star M^4 \otimes TM^4$ — the notation $\sigma_\kappa(\mathcal{S})$ means σ_κ applied to the eigenvalues of $\mathcal{S}$. In particular, given a section of the bundle of symmetric two–tensors such as A, by "raising an index" we can cannonically associate a section $g^{-1}A$ of $\text{End}(TM^4)$. At each point of M^4, $g^{-1}A$ has 4 real eigenvalues, thus $\sigma_\kappa(g^{-1}A)$ is a smooth function on M^4. To simplify notation, we denote $\sigma_\kappa(A) = \sigma_\kappa(g^{-1}A)$.

Returning to the aforementioned splitting of the Euler form, the Chern–Gauss–Bonnet formula (0.4) may be written

$$8\pi^2 \chi(M^4) = \int \frac{1}{4}|W|^2 dvol + \int \sigma_2(A)dvol. \tag{1.1}$$

Note that the conformal invariance of the Weyl tensor implies that the quantity

$$\int \sigma_2(A)dvol$$

is conformally invariant as well. Using (1.1), assumption (0.5) of Theorem A' can be expressed

$$\int \sigma_2(A) dvol - \int \frac{1}{4}|W|^2 dvol > 0 \, . \tag{1.2}$$

Our goal in this section is to prove the following:

Theorem 1.1. *Let* (M^4, g_0) *be a smooth, closed Riemannian four–manifold for which*
- (i) *The Yamabe invariant* $Y(M^4, g_0) > 0$, *and*
- (ii) *The curvature satisfies*

$$\int \sigma_2(A_{g_0}) dvol - \int \frac{\alpha}{4}|W_{g_0}|^2 dvol > 0 \, , \tag{1.3}$$

where $\alpha \geq 0$. *Then there is a conformal metric* $g_\alpha = e^{2w_\alpha} g_0$ *whose curvature satisfies*

$$\sigma_2(A_{g_\alpha}) - \frac{\alpha}{4}|W_{g_\alpha}|^2 \equiv \lambda \, , \tag{1.4}$$

where λ *is a positive constant.*

Theorem 1.1 is a refinement of Corollary B of [CGY2], which for comparison's sake we now state:

Theorem 1.2. *(see* [CGY2]*). Let* (M^4, g_0) *be a smooth, closed Riemannian four–manifold for which*
- (i) *The Yamabe invariant* $Y(M^4, g_0) > 0$,
- (ii) *The curvature satisfies*

$$\int \sigma_2(A_{g_0}) dvol_{g_0} > 0 \, . \tag{1.5}$$

Then there is a conformal metric $g = e^{2w} g_0$ *whose curvature satisfies*

$$\sigma_2(A_g) \equiv \lambda \, , \tag{1.6}$$

where λ *is a positive constant.*

The proof of Theorem 1.2 relies on a crucial preliminary result:

Theorem 1.3. *(See* [CGY1]*). Let* (M^4, g_0) *be a smooth, closed Riemannian four–manifold for which*
- (i) *The Yamabe invariant* $Y(M^4, g_0) > 0$,
- (ii) *The curvature satisfies*

$$\int \sigma_2(A_{g_0}) dvol_{g_0} > 0. \tag{1.7}$$

Then there is a conformal metric $g = e^{2w} g_0$ whose curvature satisfies

$$\sigma_2(A_g) > 0. \tag{1.8}$$

The importance of Theorem 1.3 is that the metric satisfying inequality (1.8) provides an approximate solution to equation (1.6), which can then be deformed to an actual solution. Moreover, (1.8) implies that the path of equations connecting the metric constructed in Theorem 1.3 to the metric constructed in Theorem 1.2 is elliptic.

The proof of Theorem 1.1 will parallel the proofs of Theorems 1.2 and 1.3. The first step is the following analogue of Theorem 1.3:

Theorem 1.4. *Let (M^4, g_0) be a smooth, closed Riemannian four–manifold for which*
   (i) *The Yamabe invariant $Y(M^4, g_0) > 0$,*
   (ii) *The curvature satisfies*

$$\int \sigma_2(A_{g_0}) dvol_{g_0} - \int \frac{\alpha}{4} |W_{g_0}|^2 dvol_{g_0} > 0 \tag{1.9}$$

where $\alpha \geq 0$. Then there is a conformal metric $g = e^{2w} g_0$ whose curvature satisfies

$$\sigma_2(A) - \frac{\alpha}{4} |W|^2 > 0. \tag{1.10}$$

Remark. To simplify notation, we will use subscripts with 0 instead of g_0, and denote the volume form by dv_0 instead of $dvol_{g_0}$.

Proof. Following [CGY1], we consider the following functional $F : W^{2,2}(M^4) \to \mathbf{R}$:

$$F[\omega] = \gamma_1 I[\omega] + \gamma_2 II[\omega] + \gamma_3 III[\omega]$$

where $\gamma_i = \gamma_i(L)$ are constants and

$$I[\omega] = \int 4|W_0|^2 \omega dv_0 - \left(\int |W_0|^2 dv_0 \right) \log \fint e^{4\omega} dv_0 \,,$$

$$II[\omega] = \int \omega P_0 \omega dv_0 + \int 4Q_0 \omega dv_0 - \left(\int Q_0 dv_0 \right) \log \fint e^{4\omega} dv_0 \,,$$

$$III[\omega] = 12 \left(Y[\omega] - \frac{1}{3} \int \Delta_0 R_0 \omega dv_0 \right) \,,$$

$$Y[\omega] = \int (\Delta_0 \omega + |\nabla_0 \omega|^2)^2 dv_0 - \frac{1}{3} \int R_0 |\nabla_0 \omega|^2 dv_0 \,.$$

Here P denotes the Paneitz operator:

$$P = (\Delta)^2 + d^\star \left(\frac{2}{3} Rg - 2Ric \right) d \,,$$

where d is the exterior derivative , $d^\star$ is the adjoint of d, and Q is the fourth order curvature invariant:

$$Q = \frac{1}{12} \left(-\Delta R + \frac{1}{4}R^2 - 3|E|^2 \right) \,.$$

Thus

$$Q = \frac{1}{2}\sigma_2(A) + \frac{1}{12}(-\Delta R) \,. \tag{1.11}$$

As in [CGY2], we need to introduce an additional functional, which depends on the choice of a nowhere–vanishing symmetric $(0,2)$–tensor η. We then let

$$\widetilde{I}[\omega] = \int 4|\eta|_0^2 \omega dv_0 - \left(\int |\eta|_0^2 dv_0 \right) \log \fint e^{4\omega} dv_0 \,.$$

Now consider the functional

$$F[\omega] = \widetilde{\gamma}_1 \widetilde{I}[\omega] + \gamma_1 I[\omega] + \gamma_2 II[\omega] + \gamma_3 III[\omega] \,,$$

and define the conformal invariant

$$\kappa = \widetilde{\gamma}_1 \int |\eta|_0^2 dv_0 + \gamma_1 \int |W_0|^2 dv_0 + \gamma_2 \int Q_0 dv_0 \,. \tag{1.12}$$

Following the work of [CY1], we have the following existence result for extremals of F. To make the paper as self-contained as possible, we will provide a sketch of the proof.

Theorem 1.5. *(See* [CY1] *Theorem 1.1) Let (M^4, g_0) be a compact Riemannian four–manifold. If $\gamma_2, \gamma_3 > 0$ and $\kappa < \gamma_2 8\pi^2$, then $\inf F[\omega]$ is attained by some function $\omega \in W^{2,2}(M^4)$. Moreover, the metric $g = e^{2\omega}g_0$ is smooth (see* [CGY3],[UV]*) and satisfies*

$$\widetilde{\gamma}_1|\eta|^2 + \gamma_1|W|^2 + \gamma_2 Q - \gamma_3 \Delta R = \kappa vol(g)^{-1} \,. \tag{1.13}$$

Proof. To see that $\inf F[\omega]$ is attained under the assumption that $\gamma_2, \gamma_3 > 0$ and $\kappa < \gamma_2 8\pi^2$, we employ a sharp version of the Moser-Trudinger inequality established by D.Adams [Ad]: there exists a constant $C = C(M, g_0)$ such that for all $\omega \in W^{2.2}(M, g_0)$

$$log \fint e^{4(\omega - \bar{\omega})} dv_0 \le C + \frac{1}{8\pi^2} \int (\Delta_0 \omega)^2 dv_0, \tag{1.14}$$

where $\bar{\omega} = \fint \omega dv_0$.

Define
$$U_0 = U(g_0) = \tilde{\gamma}_1 |\eta|_0^2 + \gamma_1 |W|_0^2 + \gamma Q_0 - \gamma_3 \Delta_0 R_0,$$

then
$$\int U_0 dv_0 = \kappa,$$

and we can express F as

$$F[\omega] = -\kappa log \fint e^{4(\omega - \bar{\omega})} dv_0 + 4 \int U_0(\omega - \bar{\omega}) dv_0 + \gamma_2 < P_0 \omega, \omega > +12\gamma_3 Y(\omega).$$

When $\kappa \leq 0$, we have

$$F[\omega] \geq 4 \int U_0(\omega - \bar{\omega}) dv_0 + \gamma_2 < P_0 \omega, \omega > +12\gamma_3 Y(\omega).$$

When $\kappa \geq 0$, we have from (1.14) that

$$F[\omega] \geq - C\kappa - \frac{\kappa}{8\pi^2} \int (\Delta_0 \omega)^2 dv_0$$
$$+4 \int U_0(\omega - \bar{\omega}) dv_0 + \gamma_2 < P_0 \omega, \omega > +12\gamma_3 Y(\omega).$$

Thus, for a minimizing sequence $\{\omega_l\}$, $\underline{lim} F[\omega_l] \leq F[0] = 0$. From the estimates above we conclude that for l large, ϵ small,

$$\epsilon \geq F[\omega_l] \geq - C\tilde{\kappa} + (-\frac{\tilde{\kappa}}{8\pi^2} + \gamma_2 + 12\gamma_3) \int (\Delta_0 \omega_l)^2$$
$$+ 12\gamma_3 \int |\nabla_0 \omega_l|^4 + 4 \int U_0(\omega_l - \bar{\omega}_l)$$
$$+ (\frac{2}{3}\gamma_2 - 4\gamma_3) \int R_0 |\nabla_0 \omega_l|^2 - 2\gamma_2 \int Ric_0(\nabla_0 \omega_l, \nabla_0 \omega_l)$$
$$+ 24\gamma_3 \int (\Delta_0 \omega_l) |\nabla_0 \omega_l|^2,$$

where $\tilde{\kappa} = max(\kappa, 0)$. It follows that if $\kappa \leq \gamma_2 8\pi^2$, and $\gamma_2 > 0$, $\gamma_3 > 0$, one concludes that there exists a constant $C(g_0)$ so that

$$\int (\Delta_0 \omega_l)^2 + |\nabla_0 \omega_l|^4 \leq C(g_0). \tag{1.15}$$

Since the functional F is scale-invariant, we may assume without loss of generality that $\fint \omega_l dv_0 = 0$. It follows from the Poincare inequality and (1.15) that $||\omega_l||_{L^2}$ is uniformly

bounded. Therefore, $\|\omega_l\|_{2,2}$ is bounded and a subsequence will converge weakly in $W^{2,2}$ to some $\omega \in W^{2,2}(M, g_0)$ with $F[\omega] = \inf_{\omega \in W^{2,2}} F[\omega]$.

Finally, a straightforward computation (see [BO]) shows that the extremal metric $g = e^{2\omega} g_0$ satisfies the Euler equation (1.13). $\qquad\square$

Next, let $\delta \in (0, 1]$ and choose

$$
\widetilde{\gamma}_1 = \left(-\frac{1}{2}\right) \frac{\int \sigma_2(A_0) dv_0 - \frac{\alpha}{4} \int |W_0|^2 dv_0}{\int |\eta|_0^2 dv_0} < 0 \,,
$$
$$
\gamma_1 = -\frac{\alpha}{8} \,,
$$
$$
\gamma_2 = 1 \,,
$$
$$
\gamma_3 = \frac{1}{24}(3\delta - 2) \,.
$$

With these values of $(\widetilde{\gamma}_1, \gamma_1, \gamma_2, \gamma_3)$, the conformal invariant κ defined in (1.12) is equal to zero. Thus, as long as $\gamma_3 > 0$ (i.e., $\delta > \frac{2}{3}$) the hypotheses of Theorem 1.5 hold. In particular, if $\delta = 1$ there exists an extremal metric $g_1 = e^{2\omega} g_0$, satisfying (1.13), which we rewrite using (1.11):

$$
\sigma_2(A) - \frac{\alpha}{4}|W|^2 = \frac{1}{4}\Delta R - 2\widetilde{\gamma}_1 |\eta|^2 \,. \tag{1.16}
$$

Using the minimum principle of [Gu, Lemma 1.2], this implies that the scalar curvature of g_1 is strictly positive.

For general $\delta \in (0, 1]$, the Euler equation (1.13) can be written

$$
\sigma_2(A) - \frac{\alpha}{4}|W|^2 = \frac{\delta}{4}\Delta R - 2\widetilde{\gamma}_1 |\eta|^2 \,. \tag{$\star\star$}_\delta
$$

Compare this with equation $(\star)_\delta$ of [CGY1]:

$$
\sigma_2(A) = \frac{\delta}{4}\Delta R - 2\gamma_1 |\eta|^2 \,. \tag{$\star$}_\delta
$$

Note that the only discrepancy between $(\star\star)_\delta$ and $(\star)_\delta$ is the presence of the Weyl term on the LHS of $(\star)_\delta$. The key point is that this term has the same sign as the term involving η and scales in exactly the same way. That is, if $g = e^{2w} g_0$ is a conformal metric, then

$$
|W_g|_g^2 = e^{-4w}|W_0|_0^2,
$$
$$
|\eta|_g^2 = e^{-4w}|\eta|_0^2.
$$

Consequently, the subsequent arguments of section 2–6 of [CGY1] can be carried out with only trivial modifications, as we now describe.

Fixing $\delta_0 > 0$, let

$$\mathcal{S} = \left\{ \delta \in [\delta_0, 1] \;\middle|\; \begin{array}{l} (\star\star)_\delta \text{ admits a solution with} \\ \text{positive scalar cuvature} \end{array} \right\}.$$

As we saw above, $1 \in \mathcal{S}$; thus $\mathcal{S}$ is non–empty. To verify that $\mathcal{S}$ is open we compute the linearization of $(\star\star)_\delta$, exactly as in Proposition 4.1 of [CGY2]. Again, the only relevant properties of the Weyl term and η–term in $(\star\star)_\delta$ are their scaling properties (which are the same) and their sign (ditto). Next, the estimates of section 3 in [CGY1] can be be used to show that $\mathcal{S}$ is closed. Consequently, for each $\delta > 0$, there is a conformal metric $g = g_\delta = e^{2\omega_\delta} g_0$ of positive scalar curvature satisfying $(\star\star)_\delta$.

In sections 4–6 of [CGY2] we obtain the following *a priori* estimate for solutions of $(\star)_\delta$: For fixed $p < 5$, there is a constant $C = C(p)$ such that

$$\|\omega_\delta\|_{2,p} \leq C. \tag{1.17}$$

In particular, C is independent of δ. The same estimate holds for solutions of $(\star\star)_\delta$, for the reasons explained above.

In section 7 of [CGY1] the Yamabe flow is used to show that one can perturb solutions of $(\star)_\delta$ to find metrics with $\sigma_2(A) > 0$. An analogous result is true for solutions of $(\star\star)_\delta$.

Theorem 1.6. *(See Theorem 7.1 of* [CGY1]*). Let* $g = e^{2\omega} g_0$ *be a solution of* $(\star\star)_\delta$ *with positive scalar curvature, normalized so that* $\int \omega dv_0 = 0$. *If* $\delta > 0$ *is sufficiently small, then there is a smooth conformal metric* $h = e^{2v} g$ *with*

$$\sigma_2(A_h) - \frac{\alpha}{4} |W_h|^2 > 0. \tag{1.18}$$

Proof. The proof of Theorem 1.6, like the proof of its counterpart Theorem 7.1 of [CGY1], is based on careful estimates of solutions to the Yamabe flow:

$$\frac{\partial h}{\partial t} = -\frac{1}{3} Rh,$$
$$h(0, \cdot) = g = e^{2\omega} g_0.$$

Using the estimate (1.17), we show that there is a time T_0, which only depends on the background metric g_0, such that the metric $h = h(T_0, \cdot)$ satisfies (1.18). Although the arguments are essentially the same, there are some necessary modifications of the proof of Theorem 7.1 which require explanation.

First, Propositions 7.2 and 7.4, and Lemmas 7.3 and 7.5 can all be copied without change. In the statement of Proposition 7.8 we need to make an obvious change: instead of defining

$f = \sigma_2(A) + 2\gamma_1|\eta|^2$, we define $\tilde{f} = \sigma_2(A) - \frac{\alpha}{4}|W|^2 + 2\widetilde{\gamma}_1|\eta|^2$. The conclusion of Proposition 7.8 then holds with f replaced by $\tilde{f}$.

In fact, by substituting f with $\tilde{f}$, the proof of the next proposition (7.12) is also valid. Therefore, by following the remaining arguments of section 7 we arrive at the following inequality: For fixed $s \in (4,5)$ and $t \le T_1(g_0)$,

$$\sigma_2(A) - \frac{\alpha}{4}|W|^2 \ge -2\widetilde{\gamma}_1|\eta|^2 - C_3 t^{1-\frac{4}{s}} - C_3 \delta^{\frac{1}{2}} t^{(-1+\frac{2}{s})},$$

where $C_3 = C_3(g_0)$. Since $s > 4$ and $-2\widetilde{\gamma}_1|\eta|^2 \ge C(g_0) > 0$, it follows there is a constant $C_4 = C_4(g_0) > 0$ so that for $t \le T_0 = T_0(g_0)$,

$$\sigma_2(A) - \frac{\alpha}{4}|W|^2 \ge \frac{3}{4}C_4 - C_3 \delta^{\frac{1}{2}} t^{-(1+\frac{2}{s})}. \tag{1.19}$$

Therefore, if $h = (t_0, \cdot)$, then for $\delta < \delta_0(g_0)$, (1.19) implies

$$\sigma_2(A_h) - \frac{\alpha}{4}|W_h|^2 \ge \frac{1}{2}C_4 > 0.$$

This completes the proof of Theorem 1.6, and consequently Theorem 1.4.   □

Once the existence of a metric satisfying (1.18) is established, to complete the proof of Theorem 1.1 we need to show that the techniques of [CGY2] can be applied to construct a solution of (1.4). This is actually a two step process: First, we need to establish *a priori* bounds for solutions of

$$\sigma_2(A_g) - \frac{\alpha}{4}|W_g|^2 = f > 0. \tag{1.20}$$

The second step is to apply a degree-theoretic argument showing that a metric satisfying (1.18) can be deformed to a metric satisfying (1.4). Of course, such an argument relies on the estimates established in the first step.

Proposition 1.7. *(See Main Theorem of* [CGY2]*). Let* $g = e^{2w}g_0$ *be a solution of (1.20) with positive scalar curvature, and assume* (M^4, g_0) *is not conformally equivalent to the round sphere. Then there is a constant* $C = C(g_0, \|f\|_{C^2})$ *such that*

$$\max_{M^4}\{e^w + |\nabla_0 w|\} \le C. \tag{1.21}$$

Now the estimate of [CGY2] applies to equations of the form

$$\sigma_2(A_g) = f > 0,$$

whereas (1.20) includes the Weyl term. However, the argument of [CGY2] can easily be modified to cover this case, as we now explain.

As in [CGY2] we argue by contradicition: assuming the theorem is false, then there is a sequence of solutions $\{w_\kappa\}$ of (1.20) (with f fixed) such that

$$\max_M [|\nabla_0 w_\kappa| + e^{w_\kappa}] \to \infty \quad \text{as} \quad \kappa \to \infty .$$

We then apply the blow-up argument described pages 155-156 of [CGY2]. To begin, assume that $P_\kappa \in M^4$ is a point at which $(|\nabla_0 w_\kappa| + e^{w_k})$ attains its maximum. By choosing normal coordinates $\{\Phi_\kappa\}$ centered at P_κ, we may identify the coordinate neighborhood of P_κ in M^4 with the unit ball $B(1) \subset \mathbf{R}^4$ such that $\Phi_\kappa(P_\kappa) = 0$. Given $\varepsilon > 0$, we define the dilations $T_\varepsilon \colon \mathbf{R}^4 \to \mathbf{R}^4$ by $x \mapsto \varepsilon x$, and consider the sequence $w_{\kappa,\varepsilon} = T_\varepsilon^\star w_\kappa + \log \varepsilon$. Note that

$$|\nabla_0 w_{\kappa,\varepsilon}| + e^{w_{\kappa,\varepsilon}} = \varepsilon (|\nabla_0 w_\kappa| + e^{w_\kappa}) \circ T_\varepsilon .$$

Thus, for each κ we can choose ε_κ so that

$$|\nabla_0(w_{\kappa,\varepsilon_\kappa})| + e^{w_{\kappa,\varepsilon_\kappa}} \big|_{x=0} = 1 .$$

Note that $w_{\kappa,\varepsilon_\kappa}$ is defined in $B_{\frac{1}{\varepsilon_\kappa}}(0)$, and

$$|\nabla_0(w_{\kappa,\varepsilon_\kappa)}| + e^{w_{\kappa,\varepsilon_\kappa}} \leqslant 1 \quad \text{on} \quad B_{\frac{1}{\varepsilon_\kappa}}(0) .$$

To simplify notation, let us denote $w_{\kappa,\varepsilon_\kappa}$ by w_κ. Since from now on we view $\{w_\kappa\}$ as a sequence defined on dilated balls in $\mathbf{R}^4$, there will be no danger of confusing the renormalized sequence with the original sequence. Note that $g_\kappa^\star \equiv e^{2w_\kappa} T_{\varepsilon_\kappa}^\star g_0 \equiv e^{2w_\kappa} g_0^\kappa$ satisfies

$$\sigma_2(A_{g_\kappa^\star}) - \frac{\alpha}{4}|W_{g_\kappa^\star}|^2 = f \circ T_{\varepsilon_k} . \tag{1.22}$$

Furthermore, $g_0^\kappa = T_{\varepsilon_\kappa}^\star g_0 \to ds^2$; where ds^2 is the Euclidean metric on $\mathbf{R}^4$, in $C^{2,\beta}$ on compact sets.

As in [CGY2], we now have to consider two possibilities, depending on the behavior of the exponential term $e^{w_\kappa(0)}$. However, from here on the argument is identical in its details with that of [CGY2]. The main point is that the conformal invariance of the Weyl curvature implies that the two possible limit metrics arising from the sequence $g_\kappa^\star$ satisfy the same equations as they do in [CGY2]; that is, the Weyl term in (1.22) converges to zero because Euclidean space is conformally flat. In particular, Corollary 1.3 in [CGY3] applies to equations such as (1.20), so the estimates needed to construct the limiting metric are the same.

To summarize: after applying the same blow-up argument to our sequence, we end up with the same limiting equations on $\mathbf{R}^4$ (see Corollary 1.4 in [CGY2]). The rest of the proof carries through exactly as in [CGY2], and we conclude that the manifold (M^4, g) must be

conformally equivalent to the round 4-sphere. Since this contradicts our assumption, the estimate (1.21) must hold.

The local estimate of [CGY2, Cor 1.3] also applies, and consequently we have a bound

$$\|\nabla^2 w\|_\infty \le C \,. \tag{1.23}$$

Next, we use a degree theoretic argument to prove the existence of a solution of (1.4). The following Proposition is a fairly straightforward modification of Corollary B in [CGY2].

Proposition 1.8. *Assume that (M^4, g_0) satisfies conditions (i) and (ii) of Theorem 1.4. Then given any positive (smooth) function $f > 0$, there exists a solution $g = e^{2w} g_0$ of (1.20). In particular, this is true if $f \equiv \lambda$ for a constant $\lambda > 0$.*

Proof. We first apply Theorem 1.4 to assert the existence of a conformal metric $g = e^{2w} g_0$ for which the equation (1.20) holds for some positive function f. Since by assumption (M^4, g_0) is not conformally equivalent to the standard 4-sphere, the *a priori* estimates (1.22) and (1.23) hold. In particular, given a smooth function h, there is a constant c independent of t so that all solutions $g = e^{2w} g_0$ of the equation

$$\sigma_2(A_g) - \frac{\alpha}{4}|W_g|^2 = tf + (1-t)h \tag{Σ_t}$$

with $R = R_g > 0$ satisfy the bounds

$$\|w\|_{4,\alpha} \le c, \quad S_{ij}(g)\xi_i\xi_j \ge \tfrac{1}{c}\,|\xi|^2, \tag{1.24}$$

where $S = -Ric + \frac{1}{2}Rg$ (see [CGY1], Lemma 1.2). Let O_c be the set

$$O_c = \{w \in C^{4,\alpha} : (1.24) \text{ holds }\} \cap \{w \in C^{4,\alpha} | (\sigma_2(A_{g_w} - \frac{\alpha}{4}|W_{g_w}|^2) > 0; \ R_{g_w} > 0\}.$$

We denote the degree of the equation (Σ_t) by $\deg(\Sigma_t, O_c, 0)$. The degree theory of [Li] implies that

$$deg(\Sigma_0, O_c, 0) = deg(\Sigma_1, O_c, 0). \tag{1.25}$$

We need to do a calculation verifying that for $t = 1$ the degree of the equation is non-zero. In order to do this, we deform the equation to one whose degree is easy to determine. First, it is useful to re-write equation (1.20) in a suggestive form. Suppose $g = e^{2w} g_0$ and denote

$$M_{ij}(w) = 2S_{ij}^0 + 2\nabla_i^0 \nabla_j^0 w - 2\Delta_0 w g_{ij}^0 - 2\nabla_i^0 w \nabla_j^0 w. \tag{1.26}$$

Then, after some computation, the equation (1.20) may be written in the form

$$-\nabla_i^0 \{M_{ij}(w)\nabla_j^0 w\} + (\sigma_2(A_{g_0}) - \frac{\alpha}{4}|W_{g_0}|^2) = (\sigma_2(A_g) - \frac{\alpha}{4}|W_g|^2)e^{4w} = fe^{4w}. \tag{1.27}$$

It is important to note the identity

$$M_{ij}(w) = S_{ij} + S_{ij}^0 + |\nabla_0 w|^2 g_{ij}^0,$$

so that it is clear that when both $(\sigma_2(A_g) - \frac{\alpha}{4}|W_g|^2) > 0, R_g > 0$ and $(\sigma_2(A_{g_0} - \frac{\alpha}{4}|W_{g_0}|^2) > 0, R_{g_0} > 0$, then M_{ij} is positive definite.

It is also convenient to re-formulate equation (1.20), on account of the conformal covariance property, using the solution metric g of the equation as the background metric:

$$-\nabla_i \{M_{ij}(v)\nabla_j v\} + f = fe^{4v}, \tag{1.28}$$

so that $v = 0$ is a solution to this equation satisfying $R > 0$.

We now use the following deformation:

$$-\nabla_i \{M_{ij}(v)\nabla_j v\} + f = (\sigma_2(A_{g_v}) - \frac{\alpha}{4}|W_{g_v}|^2)e^{4v} = (1-t)f \int e^{4v} + tfe^{4v}$$

where $\int e^{4v} = \int e^{4v} dvol_g$. We label this equation by Γ_t. Note that when $t = 1$, we recover the equation (1.28). The proof of the Proposition now follows line by line the proof of Corollary B in [CGY2]. More precisely, after establishing *a priori* estimates for solutions of Γ_t, we find that the degree is well defined. A calculation of the linearized equation, together with the homotopy invariance of the degree implies that $deg(\Gamma_1, O_c, 0) = deg(\Gamma_0, O_c, 0) = -1$. It follows that a solution of (1.20) exists.   □

Applying Proposition 1.8, we complete the proof of Theorem 1.1.

2. The proof of Theorem A

Based on the results of section 1, we can now give a detailed proof of Theorem A.

As we saw above, the assumption (0.3) is equivalent to the inequality (1.2):

$$\int_{M^4} \sigma_2(A)dvol - \int_{M^4} \frac{1}{4}|W|^2 dvol > 0.$$

Taking $\alpha = 1$ in Theorem 1.1, it follows that there is a conformal metric satisfying

$$\sigma_2(A) - \frac{1}{4}|W|^2 \equiv \lambda > 0. \tag{2.1}$$

Strictly speaking, at this stage all we really need is the conclusion of Theorem 1.4 — that is, we just need to know that the quantity in (2.1) is positive, not necessarily constant.

In any case, rewriting $\sigma_2(A)$ in terms of the trace–free Ricci tensor $E = Ric - \frac{1}{4}Rg$ and the scalar curvature we conclude

$$-\frac{1}{2}|E|^2 + \frac{1}{24}R^2 - \frac{1}{4}|W|^2 > 0 \,.$$

Rearranging terms, this implies

$$\frac{|W|^2 + 2|E|^2}{R^2} < \frac{1}{6} \,.$$

By the weak-pinching result of Margerin [Ma], M^4 is diffeomorphic to S^4 or $\mathbf{R}P^4$. $\square$

3. A Weitzenböck formula for Bach–flat metrics

In preparation for the proof of Theorem C, in this section we derive various curvature identities. The first such result is an inequality for metrics satisfying (1.4).

Lemma 3.1. *Suppose (M^4, g) satisfies (1.4):*

$$\sigma_2(A) - \frac{\alpha}{4}|W|^2 = \lambda \,,$$

where $\alpha \geq 0$ and $\lambda \geq 0$ are constants. Then

$$\frac{3}{2}\alpha|\nabla W|^2 + 3(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2) \geq 0 \,. \tag{3.1}$$

Proof. Define the tensor

$$V = \sqrt{\alpha}\, W + \frac{1}{2}E \owedge g \,,$$

where $\owedge$ is the Kulkarni–Nomizu product. Then

$$|V|^2 = \alpha|W|^2 + 2|E|^2 \tag{3.2}$$
$$|\nabla V|^2 = \alpha|\nabla W|^2 + 2|\nabla E|^2 \,. \tag{3.3}$$

As a consequence of (3.3), inequality (3.1) is equivalent to

$$|\nabla V|^2 \geq \frac{1}{6}|\nabla R|^2 \,. \tag{3.4}$$

To verify (3.4) note that (1.4) and (3.2) imply

$$\frac{1}{6}R^2 = |V|^2 + 4\lambda \tag{3.5}$$

Differentiating,

$$\frac{1}{3}R\nabla R = \nabla|V|^2 = 2|V|\nabla|V|\,.$$

Taking the inner product of both sides with $R^{-1}\nabla R$ gives

$$\frac{1}{3}|\nabla R|^2 = 2|V|g\left(\nabla|V|,\frac{\nabla R}{R}\right) \le |V|^2\frac{|\nabla R|^2}{R^2} + |\nabla|V||^2\,.$$

By Kato's inequality, $|\nabla|V||^2 \le |\nabla V|^2$, and thus

$$\frac{1}{3}|\nabla R|^2 \le |V|^2\frac{|\nabla R|^2}{R^2} + |\nabla V|^2\,. \tag{3.6}$$

Substituting (3.5) into (3.6) gives

$$\frac{1}{3}|\nabla R|^2 \le \left(\frac{1}{6}R^2 - 4\lambda\right)\frac{|\nabla R|^2}{R^2} + |\nabla V|^2 \le \frac{1}{6}|\nabla R|^2 + |\nabla V|^2\,,$$

because $\lambda \ge 0$. This establishes (3.4), and consequently (3.1). $\square$

We now recall two important identities for Bach-flat metrics. The first may be found in [CGY1]:

Proposition 3.2. *(See* [CGY1], *Lemma 5.4).* *If* (M^4, g) *is Bach-flat, then*

$$0 = \int\limits_{M^4}\left\{3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) + 6trE^3 + R|E|^2 - 6W_{ijk\ell}E_{ik}E_{j\ell}\right\}dvol\,, \tag{3.7}$$

where $trE^3 = E_{ij}E_{ik}E_{jk}$.

Proof. Just combine identities (5.5) and (5.10) of [CGY1]. $\square$

The second identity is a consequence of Stokes' Theorem, the Bianchi identities, and the definition of the Bach tensor in (0.7).

Proposition 3.3. *If* (M^4, g) *is Bach flat, then*

$$\int\limits_{M^4}|\nabla W|^2 dvol = \int\left\{72\det W^+ + 72\det W^- - \frac{1}{2}R|W|^2 + 2W_{ij\kappa\ell}E_{i\kappa}E_{j\ell}\right\} \tag{3.8}$$

Proof. In [De], Derdzinski proved a similar formula for metrics with harmonic Weyl tensor. Our identity differs from his in only one respect; namely, we replace harmonicity of the Weyl tensor (a first order condition) with Bach flatness (a second order condition). To simplify

the calculations and harmonize our notation with [De], we compute with respect to a local (normal) frame field. Using this convention, we write

$$\int \left|\nabla W^+\right|^2 = \int \nabla_m W^+_{ijk\ell} \nabla_m W^+_{ijk\ell}. \tag{3.9}$$

Since the splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ is parallel with respect to the Riemannian connection,

$$\nabla W = \nabla W^+ + \nabla W^-. \tag{3.10}$$

Therefore,

$$\int \left|\nabla W^+\right|^2 = \int \nabla_m W_{ijk\ell} \nabla_m W^+_{ijk\ell}. \tag{3.11}$$

Using the decomposition (1.0), the second Bianchi identity can be written

$$\begin{aligned}
0 = {} & \nabla_m W_{ijk\ell} + \nabla_i W_{jmk\ell} + \nabla_j W_{mik\ell} \\
& + \tfrac{1}{2}\big[g_{ik}(dA)_{mj\ell} - g_{i\ell}(dA)_{mjk} \\
& - g_{jk}(dA)_{mi\ell} + g_{j\ell}(dA)_{mik} - g_{km}(dA)_{ij\ell} \\
& + g_{m\ell}(dA)_{ijk}\big],
\end{aligned} \tag{3.12}$$

where

$$(dA)_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}.$$

Contracting (3.12) we get the identity

$$(\delta W)_{ij\ell} \equiv \nabla_m W_{ijm\ell} = \tfrac{1}{2}(dA)_{ij\ell}, \tag{3.13}$$

where δ denotes the divergence. Subtituting (3.12) into (3.11) we get

$$\begin{aligned}
\int \left|\nabla W^+\right|^2 = \int \nabla_m W^+_{ijk\ell} \big\{ & -\nabla_i W_{jmk\ell} \\
& - \nabla_j W_{mik\ell} + \tfrac{1}{2}\big[g_{km}(dA)_{ij\ell} - g_{m\ell}(dA)_{ijk}\big] \big\},
\end{aligned}$$

because all other terms vanish due to the symmetries of the Weyl tensor. Re–indexing and combining like terms, we find

$$\int \left|\nabla W^+\right|^2 = \int -2\nabla_m W^+_{ijk\ell} \nabla_i W_{jmk\ell} + \nabla_m W^+_{ijm\ell}(dA)_{ij\ell}. \tag{3.14}$$

The splitting (3.10) implies $\delta W = \delta W^+ + \delta W^-$, so by (3.13)

$$\nabla_m W_{ijm\ell}(dA)_{ij\ell} = 2(\delta W^+)_{ij\ell}(\delta W)_{ij\ell} = 2|\delta W^+|^2.$$

Similarly,

$$-2\nabla_m W^+_{ijk\ell}\nabla_i W_{jmk\ell} = -2\nabla_m W^+_{ijk\ell}\nabla_i W^+_{jmk\ell}.$$

Substituting these into (3.14) we obtain

$$\int |\nabla W^+|^2 = \int -2\nabla_m W^+_{ijk\ell}\nabla_i W^+_{jmk\ell} + 2|\delta W^+|^2. \tag{3.15}$$

We analyze each term in (3.15) separately.

For the first term, we integrate by parts and commute derivatives:

$$\int -2\nabla_m W^+_{ijk\ell}\nabla_i W^+_{jmk\ell} = \int 2W^+_{ijk\ell}\nabla_m\nabla_i W^+_{jmk\ell}$$
$$= \int 2W^+_{ijk\ell}\{\nabla_i\nabla_m W^+_{jmk\ell} + R_{mijs}W^+_{smk\ell}$$
$$+ R_{mims}W^+_{jsk\ell} + R_{miks}W^+_{jms\ell}$$
$$+ R_{mi\ell s}W^+_{jmks}\}$$
$$= \int 2W^+_{ijk\ell}\nabla_i\nabla_m W^+_{jmk\ell} + 2R_{mijs}W^+_{ijk\ell}W^+_{smk\ell}$$
$$+ 2R_{is}W^+_{ijk\ell}W^+_{jsk\ell} + 2R_{miks}W^+_{ijk\ell}W^+_{jms\ell}$$
$$+ 2R_{mi\ell s}W^+_{ijk\ell}W^+_{jmks}, \tag{3.16}$$

where $R_{is} = R_{mims}$ are the components of the Ricci tensor. Note by re–indexing the last two terms in (3.16) are equal. If we integrate by parts again, the first term in (3.16) is

$$\int 2W^+_{ijk\ell}\nabla_i\nabla_m W^+_{jmk\ell} = \int -2\nabla_i W^+_{ijk\ell}\nabla_m W^+_{jmk\ell} = \int 2|\delta W^+|^2. \tag{3.17}$$

Using the Bianchi identity and re–indexing the next term can be rewritten

$$2R_{mijs}W^+_{ijk\ell}W^+_{smk\ell} = R_{msij}W^+_{ijk\ell}W^+_{msk\ell}. \tag{3.18}$$

Appealing to the decomposition (1.0) once more, we get

$$2R_{mijs}W^+_{ijk\ell}W^+_{smk\ell} = W^+_{msij}W^+_{ijk\ell}W^+_{msk\ell} + 2A_{js}W^+_{jik\ell}W^+_{sik\ell}. \tag{3.19}$$

Similarly,

$$4R_{miks}W^+_{ijk\ell}W^+_{jms\ell} = 4W_{miks}W^+_{ijk\ell}W^+_{jms\ell} + 2A_{is}W^+_{ijk\ell}W^+_{jks\ell} + 2A_{km}W^+_{ijk\ell}W^+_{jmi\ell}. \tag{3.20}$$

Using the symmetries of the Weyl tensor and re–indexing we find

$$2A_{is}W^+_{ijk\ell}W^+_{jks\ell} + 2A_{km}W^+_{ijk\ell}W^+_{jmil} = -2A_{km}W^+_{ij\ell k}W^+_{ij\ell m}. \tag{3.21}$$

Combining (3.15)–(3.21),

$$\int |\nabla W^+|^2 = \int 4|\delta W^+|^2 + 2R_{is}W^+_{ijk\ell}W^+_{jsk\ell}$$
$$+ W^+_{msij}W^+_{ijk\ell}W^+_{msk\ell} + 4W_{miks}W^+_{ijk\ell}W^+_{jms\ell}. \tag{3.22}$$

Lemma 3.4.

(i) $W^+_{ijk\ell}W^+_{jsk\ell} = -\frac{1}{4}|W^+|^2\delta_{ij}$,

(ii) $W^+_{msij}W^+_{ijk\ell}W^+_{msk\ell} = 24\det W^+$,

(iii) $4W_{miks}W^+_{ijk\ell}W^+_{jms\ell} = 48\det W^+$.

Proof. As in [De], we fix a point and diagonalize $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$. Let $\lambda^\pm_i, 1 \leq i \leq 3$ denote the three eigenvalues of $W^\pm$, with corresponding eigenforms $\omega^\pm, \eta^\pm, \theta^\pm$. Then

$$W^\pm = \lambda^\pm_1\omega^\pm \otimes \omega^\pm + \lambda^\pm_2\eta^\pm \otimes \eta^\pm + \lambda^\pm_3\theta^\pm \otimes \theta^\pm.$$

Then (i)–(iii) follow from elementry calculations. $\square$

From the preceding Lemma and (3.22) we obtain the identity

$$\int |\nabla W^+|^2 = \int 4|\delta W^+|^2 + 72\det W^+ - \tfrac{1}{2}R|W^+|^2, \tag{3.23}$$

which holds for any Riemannian four–manifold.

Now suppose (M^4, g) is Bach–flat. By [De, (23)],

$$0 = \nabla_k\nabla_\ell W_{ik\ell j} - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}.$$

Pairing both sides with the Weyl–Schouten tensor and integrating we get

$$\begin{aligned}
0 &= \int A_{ij}\nabla_k\nabla_\ell W_{ik\ell j} - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}A_{ij} \\
&= \int -\nabla_k A_{ij}\nabla_\ell W_{ik\ell j} - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}A_{ij} \\
&= \int -\tfrac{1}{2}(\nabla_k A_{ij} - \nabla_i A_{kj})\nabla_\ell W_{ik\ell j} - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}A_{ij} \\
&= \int -\tfrac{1}{2}(dA)_{kij}\nabla_\ell W_{ik\ell j} - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}A_{ij} \\
&= \int |\delta W|^2 - \tfrac{1}{2}W_{ikj\ell}A_{k\ell}A_{ij}.
\end{aligned}$$

Since the Weyl tensor is trace–free, we conclude

$$\int |\delta W|^2 = \int \tfrac{1}{2}W_{ijk\ell}E_{ik}E_{j\ell}. \tag{3.24}$$

Combining (3.23) and (3.24), we get

$$\begin{aligned}
\int |\nabla W|^2 &= \int |\nabla W^+|^2 + \int |\nabla W^-|^2 \\
&= \int 72\det W^+ + 72\det W^- - \tfrac{1}{2}R|W|^2 + 2W_{ijk\ell}E_{ik}E_{j\ell}.
\end{aligned}$$

This completes the proof of Proposition 3.3. $\square$

Combining (3.1), (3.7) and (3.8) we find that for any $\alpha \geq 0$,

$$
\begin{aligned}
0 = \int_{M^4} & \left\{ \frac{3}{2}\alpha|\nabla W|^2 + 3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \right. \\
& + 6\,tr\,E^3 + R|E|^2 - 3(\alpha+2)W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \\
& \left. -108\alpha\det W^+ - 108\alpha\det W^- + \frac{3}{4}\alpha R|W|^2 \right\} dvol \\
\geq \int_{M^4} & 6\,tr\,E^3 + R|E|^2 - 3(\alpha+2)W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \\
& -108\alpha\det W^+ - 108\alpha\det W^- + \frac{3}{4}\alpha R|W|^2 \,. \qquad (3.25)
\end{aligned}
$$

This is the key identity in the proof of Theorem C.

4. The proof of Theorem C

Suppose (M^4, g) is Bach–flat with positive Yamabe invariant, and that (0.6) holds:

$$
\int_{M^4} |W|^2 dvol = 16\pi^2 \chi(M^4)\,.
$$

This is equivalent to

$$
\int_{M^4} \sigma_2(A) dvol = \frac{1}{4}\int_{M^4} |W|^2 dvol\,. \qquad (4.1)
$$

Now, if $W \equiv 0$ then (M^4, g) is locally conformally flat and (by (0.6)) $\chi(M^4) = 0$. It follows from [Gu, Corollary G] that (M^4, g) is conformal to a manifold which is isometrically covered by $S^3 \times S^1$. Therefore, let us assume from now on that

$$
\int_{M^4} |W|^2 dvol > 0\,. \qquad (4.2)
$$

By (4.1), this implies that for any $0 \leq \alpha < 1$,

$$
\int_{M^4} \sigma_2(A) dvol - \frac{\alpha}{4}\int_{M^4} |W|^2 dvol = \frac{(1-\alpha)}{4}\int_{M^4} |W|^2 dvol > 0\,.
$$

According to Theorem 1.1, there is a conformal metric $g_\alpha = e^{2\omega_\alpha} g$ satisfying

$$
\sigma_2(A_{g_\alpha}) - \frac{\alpha}{4}|W_{g_\alpha}|^2 \equiv \lambda_\alpha > 0\,. \qquad (4.3)
$$

Choose a sequence $\alpha_\kappa \nearrow 1$, and denote $g_\kappa = g_{\alpha_\kappa} = e^{2\omega_\kappa}g$. The compactness properties of the sequence $\{g_\kappa\}$ require careful description.

First, we claim that the a priori estimate of [CGY2] holds: i.e., there is a constant C such that

$$\|\nabla\omega_\kappa\|_\infty + \|\omega_\kappa\|_\infty \leq C\,. \tag{4.4}$$

(Recall (4.2) implies that (M^4, g) cannot be conformally equivalent to the sphere). This estimate is immediate from Proposition 1.7. In addition, the local estimate of [CGY2, Cor 1.3] applies, and consequently we have a bound

$$\|\nabla^2\omega_\kappa\|_\infty \leq C\,. \tag{4.5}$$

It is important to note that (4.5) is optimal: (4.3) is elliptic if $\sigma_2(A) > 0$; but $\lambda_\kappa \to 0$ as $\kappa \to \infty$, and there is no guarantee that the Weyl tensor does not vanish on M^4. However, higher order estimates for $\{g_\kappa\}$ can be established on the set where the Weyl tensor is non–zero. To explain this, let

$$M_+^4 = \{x \in M^4 : |W_g| > 0\},$$
$$M_0^4 = \{x \in M^4 : |W_g| = 0\}\,.$$

By conformal invariance, the Weyl tensor of each g_κ is also non-zero on M_+^4 (and vanishes on M_0^4). If $x_0 \in M_+^4$, then there are constants $\epsilon > 0, \rho > 0$, such that $|W_g| \geq \epsilon > 0$ on the geodesic ball $B_\rho(x_0) = \{x \in M^4 : dist_g(x, x_0) < \rho\}$. On $B_\rho(x_0)$ the metric $g_\kappa = e^{2\omega_\kappa}g$ satisfies

$$\sigma_2(A_{g_\kappa}) = \frac{\alpha_\kappa}{4}|W_{g_\kappa}|^2 + \lambda_\kappa = \frac{\alpha_\kappa}{4}e^{-4\omega_\kappa}|W_g|^2 + \lambda_\kappa\,.$$

In particular, by the a priori estimate (4.4) we see that

$$\sigma_2(A_{g_\kappa}) \geq C_\epsilon > 0$$

on $B_\rho(x_0)$. Therefore,

$$\sqrt{\sigma_2(A_{g_\alpha})} = \sqrt{\frac{\alpha}{4}|W_{g_\alpha}|^2 + \lambda_\alpha}$$

is a strictly elliptic, concave equation on $B_\rho(x_0)$. The regularity results of Evans [Ev] and Krylov [Kr] then give Hölder estimates for $\nabla^2\omega_\kappa$ on $B_{\rho'}(x_0)$ for any $\rho' < \rho$ (see [GT, Theorem 17.14]). Applying Schauder theory and classic elliptic regularity, we then obtain estimates for derivatives of all orders on any ball $B_{\rho'}(x_0) \subsetneq B_\rho(x_0)$ with $\rho' < \rho$. Consequently, a subsequence of $\{g_\kappa\}$ (also denoted $\{g_\kappa\}$) converges to a limiting metric g_∞ in $C_{loc}^\infty(M_+^4) \cap C^{1,1}(M^4)$.

Recall that inequality (3.25) is satisfied by each g_κ (with $\alpha = \alpha_\kappa$). If we split the integral in (3.25) into two integrals, one over M_+^4, the other over M_0^4, note that on M_0^4 the integrand reduces to

$$6trE^3 + R|E|^2\,.$$

Using the sharp inequality $trE^3 \geq -\frac{1}{\sqrt{3}}|E|^3$, we find that

$$6trE^3 + R|E|^2 \geq -2\sqrt{3}|E|^2 + R|E|^2 = |E|^2\left(R - 2\sqrt{3}|E|\right).\tag{4.6}$$

Since $\sigma_2(A) \geq 0$ on M_0^4, we also have

$$\begin{aligned}
0 &\leq -\tfrac{1}{2}|E|^2 + \tfrac{1}{24}R^2 \\
&\Rightarrow R^2 \geq 12|E|^2 \\
&\Rightarrow R \geq 2\sqrt{3}|E|.
\end{aligned}\tag{4.7}$$

Combining (4.6) and (4.7) we see that the integrand in (3.25) is non-negative on the set M_0^4. Therefore, in view of the convergence of $\{g_\kappa\}$ on M_+^4 we have

$$\begin{aligned}
0 \geq \lim_{\kappa\to\infty} \int_{M_+^4} & 6trE^3 + R|E|^2 - 3(\alpha_\kappa + 2)W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \\
& - 108\alpha_\kappa \det W^+ - 108\alpha_\kappa \det W^- + \frac{3}{4}\alpha_\kappa R|W|^2 \\
= \int_{M_+^4} & 6trE^3 + R|E|^2 - 9W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \\
& - 108\det W^+ - 108\det W^- + \frac{3}{4}R|W|^2.
\end{aligned}\tag{4.8}$$

To summarize: we have constructed a metric $g_\infty = e^{2w_\infty}g$ on M^4 with $w_\infty \in C^\infty(M_+^4)\cap C^{1,1}(M^4)$ which satisfies

$$\left.\begin{aligned}
\sigma_2(A) &= \tfrac{1}{4}|W|^2 \text{ on } M_+^4, \\
0 \geq \int_{M_+^4} \Big\{ &-108\det W^+ - 108\det W^- \\
&+ 6trE^3 - 9W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \\
&+ R\Big(|E|^2 + \tfrac{3}{4}|W|^2\Big)\Big\}dvol.
\end{aligned}\right\}\tag{4.9}$$

Proposition 4.1. *If g_∞ satisfies (4.9), then*
  (i) $g_\infty \in C^\infty(M^4)$;
  (ii) g_∞ *is Einstein;*
  (iii) *either $W^+ \equiv 0$ or $W^- \equiv 0$ on M^4.*

Proof. Our first task is to rewrite the integrand in (4.9) in a suitable basis. To this end, let $Riem : \Lambda^2 \to \Lambda^2$ denote the curvature operator of (M^4, g_∞). Since M^4 is oriented, we have the splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, and the well known decomposition of Singer–Thorpe [ST]:

$$Riem = \begin{pmatrix} W^+ + \frac{1}{6}R\,Id & \vdots & B \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ B^\star & \vdots & W^- + \frac{1}{6}R\,Id \end{pmatrix} \tag{4.10}$$

Note the compositions satisfy

$$BB^\star : \Lambda^2_+ \to \Lambda^2_+ \,,$$
$$B^\star B : \Lambda^2_- \to \Lambda^2_- \,. \tag{4.11}$$

Fix a point $P \in M^4_+$, and let $\lambda^\pm_1 \leq \lambda^\pm_2 \leq \lambda^\pm_3$ denote the eigenvalues of $W^\pm$. Then

$$\det W^\pm = \lambda^\pm_1 \lambda^\pm_2 \lambda^\pm_3 \tag{4.12}$$
$$|W^\pm|^2 = 4\|W^\pm\|^2 = 4[(\lambda^\pm_1)^2 + (\lambda^\pm_2)^2 + (\lambda^\pm_3)^2] \,. \tag{4.13}$$

Recall that $\|W^\pm\|$ denotes the norm of $W^\pm$ when interpreted as an endomorphism of $\Lambda^2_\pm$.

Following Margerin, we denote the eigenvalues of $BB^\star : \Lambda^2_+ \to \Lambda^2_+$ by $b^2_1 \leq b^2_2 \leq b^2_3$, where $0 \leq b_1 \leq b_2 \leq b_3$.

Lemma 4.2.

$$|E|^2 = 4(b^2_1 + b^2_2 + b^2_3), \tag{4.14}$$
$$tr\,E^3 \geq -24b_1 b_2 b_3 \,. \tag{4.15}$$

Proof. Given a basis $\{e_i\}$ of $T_p M^4$, let $\{e^\star_i\}$ denote the dual basis of $T^\star_p M^4$. Relative to this basis, the curvature operator is given by

$$Riem(e^\star_i \wedge e^\star_j) = \frac{1}{2} \sum_{\kappa, \ell} R_{ij\kappa\ell} e^\star_\kappa \wedge e^\star_\ell \,, \tag{4.16}$$

where $R_{ij\kappa\ell}$ are components of $Riem$ viewed as a $(0, 4)$–tensor; i.e., $R_{ij\kappa\ell} = Riem(e_i, e_j, e_\kappa, e_\ell)$. If $E_{ij} = E(e_i, e_j)$ are the components of the trace–free Ricci tensor, then the decomposition (0.1) implies

$$R_{ij\kappa\ell} = W_{ij\kappa\ell} + \frac{1}{2}(\delta_{i\kappa} E_{j\ell} - \delta_{i\ell} E_{j\kappa} - \delta_{j\kappa} E_{i\ell} + \delta_{j\ell} E_{i\kappa}) \\ + \frac{1}{12}R(\delta_{i\kappa}\delta_{j\ell} - \delta_{i\ell}\delta_{j\kappa}) \,, \tag{4.17}$$

where $W_{ij\kappa\ell} = W(e_i, e_j, e_\kappa, e_\ell)$.

A basis $\{e_i^\star\}$ of $T_p^\star M^4$ induces a natural orthonormal basis of $\Lambda_\pm^2$:

$$
\begin{aligned}
\omega^\pm &= \tfrac{1}{\sqrt{2}}(e_1^\star \wedge e_2^\star \pm e_3^\star \wedge e_4^\star), \\
\eta^\pm &= \tfrac{1}{\sqrt{2}}(e_1^\star \wedge e_3^\star \mp e_2^\star \wedge e_4^\star), \\
\theta^\pm &= \tfrac{1}{\sqrt{2}}(e_1^\star \wedge e_4^\star \pm e_2^\star \wedge e_3^\star).
\end{aligned}
$$

$$(4.18)$$

Now suppose the basis $\{e_i\}$ diagonalizes E:

$$
E = \begin{pmatrix}
E_{11} & & & 0 \\
& E_{22} & & \\
& & E_{33} & \\
0 & & & E_{44}
\end{pmatrix}.
$$

Using (4.11), (4.16), and (4.17), the matrix of $B : \Lambda_-^2 \to \Lambda_+^2$ relative to the basis in (4.18) is

$$
B = \begin{pmatrix}
\tfrac{1}{4}(E_{11} + E_{22} - E_{33} - E_{44}) & & 0 \\
& \tfrac{1}{4}(E_{11} + E_{33} - E_{22} - E_{44}) & \\
0 & & \tfrac{1}{4}(E_{11} + E_{44} - E_{22} - E_{33})
\end{pmatrix}.
$$

Let

$$
\begin{aligned}
\mu_1 &= \tfrac{1}{4}(E_{11} + E_{22} - E_{33} - E_{44}), \\
\mu_2 &= \tfrac{1}{4}(E_{11} + E_{33} - E_{22} - E_{44}), \\
\mu_3 &= \tfrac{1}{4}(E_{11} + E_{44} - E_{22} - E_{33}).
\end{aligned}
$$

Since E is trace–free, these can also be expressed

$$
\begin{aligned}
\mu_1 &= \tfrac{1}{2}(E_{11} + E_{22}), \\
\mu_2 &= \tfrac{1}{2}(E_{11} + E_{33}), \\
\mu_3 &= \tfrac{1}{2}(E_{11} + E_{44}).
\end{aligned}
$$

$$(4.19)$$

Consequently, in terms of $\{\mu_1, \mu_2, \mu_3\}$ the eigenvalues of $BB^\star : \Lambda_+^2 \to \Lambda_+^2$ are

$$
\begin{aligned}
b_1^2 &= \mu_1^2, \\
b_2^2 &= \mu_2^2, \\
b_3^2 &= \mu_3^2,
\end{aligned}
$$

$$(4.20)$$

Now, a simple calculation gives

$$
\begin{aligned}
8\mu_1\mu_2\mu_3 &= (E_{11} + E_{22})(E_{11} + E_{33})(E_{11} + E_{44}) \\
&= E_{11}^3 + E_{11}^2 E_{22} + E_{11}^2 E_{33} + E_{11}^2 E_{44} \\
&\quad + E_{11}E_{22}E_{33} + E_{11}E_{22}E_{44} + E_{11}E_{33}E_{44} + E_{22}E_{33}E_{44} \\
&= E_{11}^2(E_{11} + E_{22} + E_{33} + E_{44}) + \sigma_3(E_{11}, E_{22}, E_{33}, E_{44}).
\end{aligned}
$$

On the other hand, for a symmetric trace–free 4×4 matrix E, $\sigma_3(E) = \frac{1}{3}trE^3$. Thus

$$
\begin{aligned}
trE^3 &= 24\mu_1\mu_2\mu_3 \\
&\geq -24|\mu_1\mu_2\mu_3| \\
&= -24b_1b_2b_3 \,.
\end{aligned}
$$

This proves (4.15). The proof of (4.14) follows from (4.20), and will be omitted. $\qquad\square$

The next inequality follows from Lemma 6 in [Ma2]. However, as our notation and conventions are slightly different we provide some details.

Lemma 4.3.

$$
-W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} \geq -4\left[\sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2\right] \tag{4.21}
$$

Proof. This inequality is termed "decoupling of the Weyl and Ricci curvatures" by Margerin, and appropriately enough: In general $W^\pm$ and $BB^\star$ or $B^\star B$ do not commute, and therefore cannot be simultaneously diagonalized.

In any case, if we choose a basis $\{e_i\}$ of T_pM^4 which diagonalizes E as in Lemma 1, then the matrix of $W^\pm : \Lambda_\pm^2 \to \Lambda_\pm^2$ relative to the basis in (4.18) is

$$
W^\pm = \begin{pmatrix} \frac{1}{2}(W_{1212} \pm 2W_{1234} + W_{3434}) & & \star \\ & \frac{1}{2}(W_{1313} \mp 2W_{1324} + W_{2424}) & \\ \star & & \frac{1}{2}(W_{1414} \pm W_{1423} + W_{2323}) \end{pmatrix}.
$$

Therefore, if $\langle\ ,\ \rangle_{\Lambda_\pm^2}$ denotes the natural inner product induced on $\Lambda_\pm^2$ (see Remark 2 in the Introduction), then

$$
\begin{aligned}
\langle W^+, BB^\star\rangle_{\Lambda_+^2} + \langle W^-, B^\star B\rangle_{\Lambda_-^2} &= tr(W^+ \circ BB^\star + W^- \circ B^\star B) \\
&= \big\{(W_{1212} + W_{3434})\mu_1^2 + (W_{1313} + W_{2424})\mu_2^2 \\
&\quad + (W_{1414} + W_{2323})\mu_3^2\big\} \\
&= \tfrac{1}{2}(W_{1212}E_{11}E_{22} + W_{1313}E_{11}E_{33} + W_{1414}E_{11}E_{44} \\
&\quad + W_{2323}E_{22}E_{33} + W_{2424}E_{22}E_{44} + W_{3434}E_{33}E_{44}).
\end{aligned}
$$

On the other hand,

$$W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} = 2(W_{1212}E_{11}E_{22} + W_{1313}E_{11}E_{33} + W_{1414}E_{11}E_{44}$$
$$+ W_{2323}E_{22}E_{33} + W_{2424}E_{22}E_{44} + W_{3434}E_{33}E_{44}).$$

Therefore,

$$-W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} = -4\langle W^+, BB^\star\rangle_{\Lambda_+^2} - 4\langle W_1^- B^\star B\rangle_{\Lambda_-^2} . \tag{4.22}$$

According to Lemma 6 of [Ma],

$$\langle W^+, BB^\star\rangle_{\Lambda_+^2} + \langle W^-, B^\star B\rangle_{\Lambda_-^2} \le \sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2. \tag{4.23}$$

Combining (4.22) and (4.23) we obtain (4.21).   □

On the set M_+^4, g_∞ satisfies

$$\sigma_2(A) = \tfrac{1}{4}|W|^2 = \|W\|^2 = \|W^+\|^2 + |W^-\|^2.$$

Therefore,

$$R = \sqrt{12|E|^2 + 24\|W^+\|^2 + 24\|W^-\|^2}. \tag{4.24}$$

Combining (4.21) and (4.24), the integrand in (4.9) at the point P satisfies the inequality

$$-108 \det W^+ - 108 \det W^- + 6tr E^3 - 9W_{ij\kappa\ell}E_{i\kappa}E_{j\ell} + R\big(|E|^2 + \tfrac{3}{4}|W|^2\big)$$
$$\ge -108\lambda_1^+\lambda_2^+\lambda_3^+ - 108\lambda_1^-\lambda_2^-\lambda_3^- - 144b_1b_2b_3$$
$$- 36\big[\lambda_1^+ b_1^2 + \lambda_2^+ b_2^2 + \lambda_3^+ b_3^2 + \lambda_1^- b_1^2 + \lambda_2^- b_2^2 + \lambda_3^- b_3^2\big]$$
$$+ \Big\{48\big(b_1^2 + b_2^2 + b_3^2\big) + 24\big[\big(\lambda_1^+\big)^2 + \big(\lambda_2^+\big)^2 + \big(\lambda_3^+\big)^2 + \big(\lambda_1^-\big)^2 + \big(\lambda_2^-\big)^2 + \big(\lambda_3^-\big)^2\big]\Big\}^{\frac{1}{2}}$$
$$\times \Big\{4\big(b_1^2 + b_2^2 + b_3^2\big) + 3\big[\big(\lambda_1^+\big)^2 + \big(\lambda_2^+\big)^2 + \big(\lambda_3^+\big)^2 + \big(\lambda_1^-\big)^2 + \big(\lambda_2^-\big)^2 + \big(\lambda_3^-\big)^2\big]\Big\}$$
$$\equiv F\big(\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_1^-, \lambda_2^-, \lambda_3^-, b_1, b_2, b_3\big) .$$

Proposition 4.4. *Suppose $0 \le b_1 \le b_2 \le b_3$, $\lambda_1^\pm \le \lambda_2^\pm \le \lambda_3^\pm$ with $\sum_{i=1}^3 \lambda_i^\pm = 0$ and $\sum_{i=1}^3\big(|\lambda_i^+|^2 + |\lambda_i^-|^2\big) \ne 0$. Then $F(\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_1^-, \lambda_2^-, \lambda_3^-, b_1, b_2, b_3) \ge 0$, and equality holds if and only if one of the following is true:*

(1) *$b_1 = b_2 = b_3 = 0$ and there exists some $a \ge 0$ with $\lambda_1^+ = \lambda_2^+ = -a$, $\lambda_3^+ = 2a$, $\lambda_1^- = \lambda_2^- = \lambda_3^- = 0$; or $\lambda_1^+ = -2a$, $\lambda_2^+ = \lambda_3^+ = a$, $\lambda_1^- = \lambda_2^- = \lambda_3^- = 0$; or similar cases with the role of λ_i^+ and λ_i^- interchanged.*
(2) *$b_1 = b_2 = b_3$, $\lambda_i^\pm = 0$ for all $1 \le i \le 3$.*

The proof of Proposition 4.4 is given in the Appendix, and amounts to a complicated Lagrange-multiplier problem. We will assume the result for now, and explain how Proposition 4.1 follows.

By Proposition 4.4, the integrand in (4.9) is non-negative. Since the integral is less than or equal to zero, it follows that the integrand $F(\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_1^-, \lambda_2^-, \lambda_3^-, b_1, b_2, b_3) \equiv 0$. Thus, at each point in M_+^4 either case (1) or case (2) of Proposition 4.4 above must hold. Since by definition $|W| > 0$ on M_+^4, case (1) is the only possibility. In particular, $E \equiv 0$ on M_+^4 and at each point either $W^+ = 0$ or $W^- = 0$.

Since $E \equiv 0$ on M_+^4 the scalar curvature is constant on each component of M_+^4, which implies by (4.9) that $|W|^2$ is also constant on each component. We claim that $M_+^4 = M^4$; i.e., M_0^4 is empty. To see why, choose a component O of M_+^4 and a sequence of points $\{x_i\}$ in O with $x_i \to x_0 \in M_0^4$. Since $|W|^2$ is constant in O,

$$c = |W_{g_\infty}|^2(x_i)$$

for some $c > 0$. By conformal invariance of the Weyl tensor,

$$c = |W_{g_\infty}|^2(x_i) = e^{-4w_\infty(x_i)}|W_g|^2(x_i).$$

By definition, $|W_g|^2(x_0) = 0$, and consequently $w_\infty(x_i) \to -\infty$ as $i \to \infty$. But this contradicts the fact that $w_\infty \in C^{1,1}$. It follows that the Weyl tensor cannot vanish on M^4, so (M^4, g_∞) is a smooth Einstein manifold. Moreover, since $|W|^2$ is constant and either $W^+ = 0$ or $W^- = 0$ at each point, it follows that one of the components of the Weyl tensor vanishes identically on M^4. By Hitchin's classification result [Hi], (M^4, g_∞) is homothetically isometric to $\pm \mathbf{C}P^2$ with the Fubini-Study metric. This completes the proof of Proposition 4.1.

Appendix

In this appendix we establish Proposition A below, which is slightly more general than Proposition 4.4.

Denote $\vec{B} = (b_1, b_2, b_3)$, $\vec{X} = (x_1, x_2, x_3)$, $\vec{Y} = (y_1, y_2, y_3)$ vectors in $\mathbb{R}^3$, and $|\vec{B}|^2 = \sum_{i=1}^3 b_i^2$, $|\vec{X}|^2 = \sum_{i=1}^3 x_i^2$, $|\vec{Y}|^2 = \sum_{i=1}^3 y_i^2$. Define the functional $I = I(\vec{B}, \vec{X}, \vec{Y})$ by

$$\begin{aligned}
I = \sqrt{6}\, &\left[4|\vec{B}|^2 + 3\left(|\vec{X}|^2 + |\vec{Y}|^2\right)\right]\left(2|\vec{B}|^2 + |\vec{X}|^2 + |\vec{Y}|^2\right)^{1/2} \\
&- 54x_1 x_2 x_3 - 54 y_1 y_2 y_3 - 72 b_1 b_2 b_3 \\
&- 18\left(x_1 b_1^2 + x_2 b_2^2 + x_3 b_3^2 + y_1 b_1^2 + y_2 b_2^2 + y_3 b_3^2\right).
\end{aligned}$$

Proposition A. *Assume* $b_1^2 \le b_2^2 \le b_3^2, \sum_{i=1}^3 x_i = 0, \sum_{i=1}^3 y_i = 0$. *Then* $I \ge 0$. *Further,* $I = 0$ *only at the following points:*
   (i) $\vec{B} = \vec{X} = \vec{Y} = (0, 0, 0)$,   *or*

(ii) $\vec{B} = (0,0,0)$ *and either* $\vec{X} = (-a,-a,2a)$, $\vec{Y} = (0,0,0)$, *or a permutation of* $x_1 = -a, x_2 = -a, x_3 = 2a$ *and* $\vec{Y} = (0,0,0)$, *or with the preceding values with the roles of* $\vec{X}$ *and* $\vec{Y}$ *reversed for some* $a \neq 0$, *or*

(iii) $\vec{B} = (b,b,b)$ *for some* $b \neq 0$ *and* $\vec{X} = \vec{Y} = (0,0,0)$.

Remark. If we set $F = 2I$, $x_i = \lambda_i^+$, $y_i = \lambda_i^-$ $(1 \leq i \leq 3)$, with $|\vec{X}|^2 + |\vec{Y}|^2 \neq 0$, then Proposition 4.4 is a consequence of Proposition A.

We will first establish Proposition A in the special case where $\vec{B} = (0,0,0)$.

Lemma 1. *Denote*

$$J(\vec{X}, \vec{Y}) = I(0, \vec{X}, \vec{Y})$$
$$= 3\sqrt{6}(|\vec{X}|^2 + |\vec{Y}|^2)(|\vec{X}|^2 + |\vec{Y}|^2)^{1/2}$$
$$- 54x_1x_2x_3 - 54y_1y_2y_3$$

with $\sum_{i=1}^3 x_i = \sum_{i=1}^3 y_i = 0$. *Then* $J \geq 0$, *and* $J = 0$ *only when* $\vec{X} = \vec{Y} = (0,0,0)$ *or at the points*

$$\vec{X} = (-a,-a,2a), \quad \vec{Y} = (0,0,0), \ or$$
$$\vec{X} = (2a,-a,-a), \quad \vec{Y} = (0,0,0), \ or$$
$$\vec{X} = (-a,2a,-a), \quad \vec{Y} = (0,0,0);$$

or with the roles of $\vec{X}$ *and* $\vec{Y}$ *reversed, for some* $a \neq 0$.

Proof. In the case $\vec{X} = \vec{Y} = (0,0,0)$, $J = 0$. In the case $|\vec{X}|^2 + |\vec{Y}|^2 \neq 0$, replacing x_i and y_i by $\frac{\pm x_i}{(|x|^2+|y|^2)^{1/2}}$ and $\frac{\pm y_i}{(|x|^2+|y|^2)^{1/2}}$, we may assume w.l.o.g. that $|\vec{X}|^2 + |\vec{Y}|^2 = 1$, $x_1 \leq x_2 \leq x_3$, and $y_1 \leq y_2 \leq y_3$.

To find the minimal points of J we use the method of Lagrange multipliers. Let $\varphi_1 = |\vec{X}|^2 + |\vec{Y}|^2 = 1$, $\varphi_2 = x_1 + x_2 + x_3 = 0$, and $\varphi_3 = y_1 + y_2 + y_3 = 0$ denote the constraints, and let $\mu, 2\beta, 2\gamma$ denote the respective Lagrange multipliers. Set $\frac{\partial J}{\partial x_i} = \mu \frac{\partial \varphi_1}{\partial x_i} + 2\beta \frac{\partial \varphi_2}{\partial x_i}$, $\frac{\partial J}{\partial y_i} = \mu \frac{\partial \varphi_1}{\partial y_i} + 2\gamma \frac{\partial \varphi_3}{\partial y_i}$ for $i = 1,2,3$; we get the equations

(A1) $-27\, x_2 x_3 = \mu\, x_1 + \beta$

(A2) $-27\, x_1 x_3 = \mu\, x_2 + \beta$

(A3) $-27\, x_1 x_2 = \mu\, x_3 + \beta$

Subtracting (A2) from (A1) we get

$$-27(x_2 - x_1)x_3 = \mu(x_1 - x_2),$$

so we have either $x_1 = x_2$, or $x_1 \neq x_2$ while $27x_3 = \mu$. Subtracting (A3) from (A2) we get similarly $x_2 = x_3$ or $x_2 \neq x_3$ while $27x_1 = \mu$. Thus, we have three possibilities: either $x_1 = x_2 = x_3 = 0$, or $x_1 = x_2 \neq x_3$ with $27x_3 = \mu$, or $x_1 \neq x_2 = x_3$ with $27x_1 = \mu$. In summary, we have $\vec{X} = (0,0,0)$ or $\vec{X} = (-a, -a, 2a)$ with $54a = \mu$; or $\vec{X} = (2a, -a, -a)$ with $54a = \mu$, where $a \geq 0$. By symmetry, we also have either $\vec{Y} = (0,0,0)$, or $\vec{Y} = (-c, -c, 2c)$ with $54c = \mu$, or $\vec{Y} = (-2c, c, c)$ with $-54c = \mu$, where $c \geq 0$.

Combining the possibilities for $\vec{X}$ and $\vec{Y}$, we have eight cases.
- (i) $\vec{X} = (0,0,0)$, $\vec{Y} = (-c, -c, 2c)$;   then $J = 0$.
- (ii) $\vec{X} = (0,0,0)$, $\vec{Y} = (-2c, c, c)$;   then $J = 0$.
- (iii) $\vec{X} = (-a, -a, 2a)$, $\vec{Y} = (-c, -c, 2c)$ with $54a = \mu = 54c \geq 0$;   then $J(\vec{X}, \vec{Y}) = 3\sqrt{6} \cdot 12a^2 \sqrt{12}\, a - 54 \cdot 4a^3 = 216(\sqrt{2} - 1)a^3 \geq 0$.
- (iv) $\vec{X} = (-a, -a, 2a)$, $\vec{Y} = (-2c, c, c)$ with $54a = \mu = -54c$;   then $a = -c$ while both $a \geq 0, c \geq 0$; thus $\vec{X} = \vec{Y} = (0,0,0)$ and $J = 0$.

Cases (v) $\rightarrow$ (viii) are similar to cases (ii) to (vi) with the roles of $\vec{X}$ and $\vec{Y}$ reversed. This finishes the proof of Lemma 1. $\square$

We now consider the general case in Proposition A. The proof is more tedious, but follows the same pattern of the proof of Lemma 1. We first outline the steps.

Outline of the proof of Proposition A.. When $\vec{B} = (0,0,0)$, we apply Lemma 1. So assume $\vec{B} \neq (0,0,0)$. We also assume w.l.o.g. that $2|\vec{B}|^2 + |\vec{X}|^2 + |\vec{Y}|^2 = 1$. To locate the minimal points of I under this constraint, we once again apply the method of Lagrange multipliers. We will break the proof into the following four steps:

<u>Step 1</u>. We may assume w.l.o.g. that $0 \leq b_1 \leq b_2 \leq b_3$ and $x_1 \leq x_2 \leq x_3$, $y_1 \leq y_2 \leq y_3$, and $x_3 \geq 0, y_3 \geq 0$.

<u>Step 2</u>. Actually, $b_1 = b_2 = b$.

<u>Step 3</u>. When $b = 0$, then $x_1 = x_2 = y_1 = y_2$ and $I > 0$.

<u>Step 4</u>. When $b > 0$, then $x_1 = x_2 = y_1 = y_2 = -a$, and either $a = 0$, $b_3 = b$, and $I = 0$;   or $a > 0$ and $I > 0$.

We now prove each of the steps in more detail.

Proof of Step 1. We first observe that at a minimal point of $I = I(\vec{B}, \vec{X}, \vec{Y})$ we may assume $b_1 b_2 b_3 \geq 0$. Thus by switching b_i with $-b_i$ $(i = 1, 2, 3)$, we may assume under the hypothesis $b_1^2 \leq b_2^2 \leq b_3^2$ that $0 \leq b_1 \leq b_1 \leq b_3$.

Next observe for any $\vec{B}$, for $\vec{X}$ to be a minimal point of I we must have $x_1 b_1^2 + x_2 b_2^2 + x_3 b_3^2 \leq x_2 b_1^2 + x_1 b_2^2 + x_3 b_3^2$. Therefore, $(x_1 - x_2)b_1^2 \leq (x_1 - x_2)b_2^2$. Thus unless $b_1^2 = b_2^2 = 0$, we have $x_1 \leq x_2$; but when $b_1^2 = b_2^2 = 0$, we may also switch the order of x_1 and x_2 if necessary, and assume $x_1 \leq x_2$ to attain the same value of I. Thus we can argue similarly and obtain

$x_1 \le x_2 \le x_3$ and $y_1 \le y_2 \le y_3$. Since $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = 0$, it follows that $x_3 \ge 0, y_3 \ge 0$.

Proof of Step 2. We now set up the Lagrange multiplier problem under the constraints

$$\varphi_1 = 2|\vec{B}|^2 + |\vec{X}|^2 + |\vec{Y}|^2 = 1$$
$$\varphi_2 = x_1 + x_2 + x_3 = 0$$
$$\varphi_3 = y_1 + y_2 + y_3 = 0$$

with multipliers $\mu, 2\beta, 2\gamma$, respectively. To locate the minimal point(s) of I under the given constraints we set $\frac{\partial I}{\partial b_i} = \mu \frac{\partial \varphi_1}{\partial b_i}$, $\frac{\partial I}{\partial x_i} = \mu \frac{\partial \varphi_1}{\partial x_i} + 2\beta \frac{\partial \varphi_2}{\partial x_i}$, $\frac{\partial I}{\partial y_i} = \mu \frac{\partial \varphi_1}{\partial y_i} + 2\gamma \frac{\partial \varphi_3}{\partial y_i}$ for $i = 1, 2, 3$ and obtain the following nine equations:

(A4) $-9(x_1 + y_1)\,b_1 - 18 b_2 b_3 = (\mu - 2\sqrt{6})b_1$

(A5) $-9(x_2 + y_2)\,b_2 - 18 b_1 b_3 = (\mu - 2\sqrt{6})b_2$

(A6) $-9(x_3 + y_3)\,b_3 - 18 b_1 b_2 = (\mu - 2\sqrt{6})b_3$

(A7) $-27(x_2 x_3) - 9b_1^2 = (\mu - 3\sqrt{6})x_1 + \beta$

(A8) $-27(x_1 x_3) - 9b_2^2 = (\mu - 3\sqrt{6})x_2 + \beta$

(A9) $-27(x_1 x_2) - 9b_3^2 = (\mu - 3\sqrt{6})x_3 + \beta$

and (A10), (A11), and (A12) are obtained by substituting y_i in place of x_i ($i = 1, 2, 3$) and γ in place of β in equations (A7), (A8), (A9).

We now assume $0 \le b_1 \le b_2 \le b_3$, $x_1 \le x_2 \le x_3$, $y_1 \le y_2 \le y_3$, and prove that (A4)—(A12) imply $b_1 = b_2$. To see this, first suppose $b_1 = 0$. Then by (A4), $b_2 b_3 = 0$; since $b_2 \le b_3$, $b_2 = 0$ and thus $b_1 = b_2$.

If $b_1 \ne 0$, then $b_2 \ne 0$, $b_3 \ne 0$. Subtracting (A8) from (A7) we get

(A13) $-27 x_3 (x_2 - x_1) - 9(b_1^2 - b_2^2) = (\mu - 3\sqrt{6})(x_1 - x_2).$

Subtracting (A11) from (A10) we get

(A14) $-27 y_3 (y_2 - y_1) - 9(b_1^2 - b_2^2) = (\mu - 3\sqrt{6})(y_1 - y_2).$

Adding (A13) and (A14),

$$-18(b_1^2 - b_2^2) = (\mu - 3\sqrt{6})(x_1 + y_1 - x_2 - y_2)$$
(A15) $$+ 27\big(x_3(x_2 - x_1) + y_3(y_2 - y_1)\big).$$

Subtracting (A5) from (A4) and substituting into (A15), we obtain

(A16) $-18(b_1^2 - b_2^2) = 2(\mu - 3\sqrt{6})\left(-\dfrac{b_2 b_3}{b_1} + \dfrac{b_1 b_3}{b_2} \right) + 27P$

where $P = x_3(x_2 - x_1) + y_3(y_2 - y_1)$. Observe that $P \geq 0$.

Thus if $b_2^2 - b_1^2 > 0$, we may divide (A16) by $b_2^2 - b_1^2$ and get

$$(A17) \qquad\qquad 18 = -2(\mu - 3\sqrt{6})\frac{b_3}{b_1 b_2} + 27\frac{P}{b_2^2 - b_1^2}.$$

Substituting (A17) into (A6), we get

$$-9(x_3 + y_3) = (\mu - 3\sqrt{6}) + \sqrt{6} + 18\frac{b_1 b_2}{b_3}$$

$$(A18) \qquad\qquad = 9\frac{b_1 b_2}{b_3} + \frac{27}{2}\frac{P}{b_2^2 - b_1^2}\frac{b_1 b_2}{b_3} + \sqrt{6}.$$

The left–hand side of (A18) is ≤ 0, while the right–hand side is $\geq \sqrt{6} > 0$. Since this contradicts the hypothesis $b_2^2 - b_1^2 > 0$, we must have $b_2^2 = b_1^2$ and hence $b_1 = b_2$. This establishes step 2.

Proof of Step 3. Denote $b_1 = b_2 = b$, assume $b = 0$, and $b_3 > 0$.

We rewrite (A6),(A13) and (A14) in this case and get

$$(A6)' \qquad\qquad -9(x_3 + y_3) = (\mu - 2\sqrt{6}),$$
$$(A13)' \qquad\qquad -27x_3(x_2 - x_1) = (\mu - 3\sqrt{6})(x_1 - x_2),$$
$$(A14)' \qquad\qquad -27y_3(y_2 - y_1) = (\mu - 3\sqrt{6})(y_1 - y_2).$$

We observe that from (A6)' (and $x_3 \geq 0$, $y_3 \geq 0$) that $\mu - 2\sqrt{6} \leq 0$. Thus we conclude from (A13)' and (A14)' that $x_1 = x_2$ and $y_1 = y_2$. Denote $x_1 = x_2 = -a$, $y_1 = y_2 = -c$, with $a, c \geq 0$. Then $\vec{B} = (0, 0, b_3)$, $\vec{X} = (-a, -a, 2a)$, $\vec{Y} = (-c, -c, 2c)$. We now assert that $a = c$. To see this, we first subtract (A9) from (A8) to get

$$(A19) \qquad\qquad -27x_1(x_3 - x_2) + 9b_3^2 = (\mu - 3\sqrt{6})(x_2 - x_3)$$

$$\qquad\qquad\qquad \text{or}$$

$$(A19)' \qquad\qquad 27a^2 + 3b_3^2 = -(\mu - 3\sqrt{6})a.$$

Similarly we subtract (A12) from (A11) and get

$$(A20)' \qquad\qquad 27c^2 + 3b_3^2 = -(\mu - 3\sqrt{6})c.$$

Subtracting (A20)' from (A19)', we then have

$$27(a^2 - c^2) = -(\mu - 3\sqrt{6})(a - c).$$

Thus, either $a = c$, or $a \neq c$ while

(A21) $$-27(a + c) = \mu - 3\sqrt{6}\,.$$

We will now derive a contradiction to see that the latter possibility ($a \neq c$) does not happen. Comparing (A21) with (A6)$'$, we see $-18(a + c) = \mu - 2\sqrt{6}$. Combining (A21) with (A6)$'$ we get

(A22) $$a + c = \frac{\sqrt{6}}{9}\,.$$

We now add (A19)$'$ to (A20)$'$ and substitute the relation $\varphi_1 = 2b_3^2 + 6(a^2 + c^2) = 1$ and (A21) into the equation to get

(A23) $$3(a^2 + c^2) = 9(a + c)^2 - 1.$$

By (A22), this expression equals $\frac{6}{9} - 1 < 0$, which is a contradiction. Thus we conclude that $a = c$ and $\vec{B} = (0, 0, b_3)$, $\vec{X} = \vec{Y} = (-a, -a, 2a)$ for some $b_3 > 0$ and $a \geq 0$. At this moment, we can check directly that $I = I(\vec{B}, \vec{X}, \vec{Y}) > 0$ if $b_3 > 0$. To be more precise, one can check that $I = 8(I_1 - I_2)$ with $I_1 = \sqrt{3}(b_3^2 + 9a^2)\,(b_3^2 + 6a^2)^{1/2}$, $I_2 = 9a(3a^2 + b_3^2)$, and $I_1^2 - I_2^2 > 0$. This establishes step 3.

Proof of Step 4. Assume $b_1 = b_2 = b \neq 0$, $b_3 > 0$. In this case, we may write (A4), (A5), (A6) as

(A4)$''$ $$-9(x_1 + y_1) - 18b_3 = \mu - 2\sqrt{6},$$
(A5)$''$ $$-9(x_2 + y_2) - 18b_3 = \mu - 2\sqrt{6},$$
(A6)$''$ $$-9(x_3 + y_2)b_3 - 18b^2 = (\mu - 2\sqrt{6})b_3\,.$$

Subtracting (A5)$''$ from (A4)$''$, we get

(A24) $$x_1 + y_1 = x_2 + y_2\,.$$

Subtracting (A8) from (A7) and (A11) from (A10) we get

(A13)$''$ $$-27x_3(x_2 - x_1) = (\mu - 3\sqrt{6})(x_1 - x_2),$$
(A14)$''$ $$-27y_3(y_2 - y_1) = (\mu - 3\sqrt{6})(y_1 - y_2)\,.$$

We now make two claims.

<u>Claim 1</u>: $x_1 = x_2(= -a)$, $y_1 = y_2(= -c)$.

<u>Claim 2</u>: $a = c$.

Proof of Claim 1. By (A13)$''$, we have either $x_1 = x_2$ or $x_1 \neq x_2$ and $27x_3 = \mu - 3\sqrt{6}$. By (A14)$''$, we have either $y_1 = y_2$ or $y_1 \neq y_2$ and $27y_3 = \mu - 3\sqrt{6}$. By (A24), we have $x_1 = x_2$ implies $y_1 = y_2$. Thus, we either have $x_1 = x_2$ and $y_1 = y_2$ as claimed, or $x_1 \neq x_2$, $y_1 \neq y_2$ while $x_3 = y_3 = \frac{1}{27}(\mu - 3\sqrt{6})$. But in the latter case, from (A6)$''$ we would have

$$-9(x_3 + y_3)b_3 - 18b^2 = (\mu - 2\sqrt{6})b_3 = (27x_3 + \sqrt{6})b_3 \,.$$

This is a contradiction as the left–hand side of the equation is less than zero, while the right–hand side is bigger than zero.

Proof of Claim 2. We follow the same strategy as in the proof of Step 3. Subtracting (A9) from (A8) we get

$$(A19)'' \qquad\qquad 27a^2 + 3(b_3^2 - b^2) = -a(\mu - 3\sqrt{6}) \,.$$

Similarly, if we subtract (A12) from (A11) we get

$$(A20)'' \qquad\qquad 27c^2 + 3(b_3^2 - b^2) = -c(\mu - 3\sqrt{6}) \,.$$

Finally, subtracting (A20)$''$ from (A19)$''$ we have

$$27(a^2 - c^2) = -(\mu - 3\sqrt{6})(a - c) \,.$$

Thus, either $a = c$ as claimed, or

$$(A21)'' \qquad\qquad a \neq c \quad \text{and} \quad 27(a + c) = -(\mu - 3\sqrt{6}) \,.$$

We will now show that (A21)$''$ cannot be true. To see this, denote $a + c = \ell \, (\ell \geq 0)$, $a - c = d$. Then $d \neq 0$, and w.l.o.g. we may assume $d > 0$. Substituting (A21)$''$ into (A4) we get

$$-18b_3 = (\mu - 2\sqrt{6}) - 9(a + c)$$
$$(A24) \qquad\qquad = (\mu - 3\sqrt{6}) - 9\ell + \sqrt{6} = -36\ell + \sqrt{6} \,.$$

Substituting (A21)$''$ to (A6) we get

$$-18b^2 = (\mu - 2\sqrt{6})b_3 + 18(a + c)b_3$$
$$(A25) \qquad\qquad = (-9\ell + \sqrt{6})b_3 \,.$$

Thus in particular

$$(A25)'' \qquad\qquad -9\ell + \sqrt{6} < 0 \,.$$

Combining (A24) and (A25), we get

(A26) $$18(b^2 - b_3^2) = (-36\ell + \sqrt{6} + 9\ell - \sqrt{6})b_3 = -27\ell b_3 \,.$$

On the other hand, substituting (A21)'' into (A19)'', we get

$$27a^2 + 3(b_3 - b^2) = 27a(a + c),$$

so

(A27) $$3(b_3^2 - b^2) = 27ac = 27\frac{\ell^2 - d^2}{4} \,.$$

Combining (A24), (A26), (A27), we get

(A28) $$27d^2 = (-9\ell + \sqrt{6})\ell,$$

which contradicts (A25)''. We conclude $a = c$, as in Claim 2.

   We are now in the situation where $\vec{B} = (b, b, b_3)$, with $b_3 \geq b > 0$, and $\vec{X} = \vec{Y} = (-a, -a, 2a)$. There are two final possibilities to consider, depending on the sign of a.

<u>Claim 3</u> If $a = 0$, then $b_3 = b \neq 0$, and $I \equiv 0$.

<u>Claim 4</u> If $a > 0$ then $I > 0$.

Proof of Claim 3. When $a = 0$, we multiply (A4)'' by b_3 then subtract (A6)'' to get $b_3^2 = b^2$, hence $b_3 = b$. In this case $I = 72b^3 - 72b^3 \equiv 0$.

Proof of Claim 4. When $a \neq 0$, we will show that $I > 0$. First, we rewrite (A4)'', (A6)'', (A7) and (A9) as follows:

(A4)'' $$18a - 18b_3 = \mu - 2\sqrt{6},$$
(A6)'' $$-36ab_3 - 18b^2 = (\mu - 2\sqrt{6})b_3,$$
(A7)'' $$54a^2 - 9b^2 = -(\mu - 3\sqrt{6})a + \beta,$$
(A9)'' $$-27a^2 - 9b_3^2 = 2(\mu - 3\sqrt{6})a + \beta \,.$$

Multiplying (A4)'' by b_3 and subtracting (A6)'' from the result we get

(A29) $$b_3^2 - b^2 = 3ab_3 \,.$$

Subtracting (A9)'' from (A7)'' we get

(A30) $$81a^2 + 9(b_3^2 - b^2) = -3a(\mu - 3\sqrt{6}) \,.$$

Combining (A29) and (A30), we get (for $a \neq 0$)

$$\text{(A31)} \qquad\qquad 27a + 9b_3 = -(\mu - 3\sqrt{6}).$$

Combining (A4)$''$ and (A31) we find

$$\text{(A32)} \qquad\qquad 45a - 9b_3 = \sqrt{6}.$$

Substituting (A29) into the constraint $\varphi_1 = 2(2b^2 + b_3^2) + 12a^2 = 1$ we get

$$\text{(A33)} \qquad\qquad 6b_3^2 - 12ab_3 + 12a^2 = 1.$$

We now introduce the notation $s = a - b_3$ and rewrite (A32) and (A33) as

$$\text{(A32)}' \qquad\qquad 36a + 9s = \sqrt{6},$$
$$\text{(A33)}' \qquad\qquad 6(s^2 + a^2) = 1.$$

Applying (A29) we can write I as

$$
\begin{aligned}
I &= \left(4(2b^2 + b_3^2) + 36a^2\right)\sqrt{6} - 216a^3 - 72b^2 b_3 - 72a(b_3^2 - b^2) \\
&= 12\sqrt{6}(b_3^2 - 2ab_3 + 3a^2) - 72(3a^3 + b_3^3 - 3a^2 b_3 + 3ab_3^2) \\
&= 72\left[\frac{s^2 + 2a^2}{\sqrt{6}} - (4a^3 - s^3)\right].
\end{aligned}
$$

It remains to check numerically that a solution (a, s) with $a > 0$ of equations (A32)$'$ and (A33)$'$ is given by $a = 0.1617...$, $s = -0.3746...$, and $I = 72(0.079... - 0.069...) > 0$.

We have thus finished the proof of Step 4 and completed the proof of Proposition A.

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